

Universality in Two-Dimensional Enhancement Percolation

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Abstract

We consider a type of dependent percolation introduced in [2], where it is shown that certain “enhancements” of independent (Bernoulli) percolation, called essential, make the percolation critical probability strictly smaller. In this paper we first prove that, for two-dimensional enhancements with a natural monotonicity property, being essential is also a necessary condition to shift the critical point. We then show that (some) critical exponents and the scaling limit of crossing probabilities of a two-dimensional percolation process are unchanged if the process is subjected to a monotonic enhancement that is not essential. This proves a form of universality for all dependent percolation models obtained via a monotonic enhancement (of Bernoulli percolation) that does not shift the critical point. For the case of site percolation on the triangular lattice, we also prove a stronger form of universality by showing that the *full* scaling limit [12, 13] is not affected by any monotonic enhancement that does not shift the critical point.

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1 Introduction

One of the most interesting phenomena in statistical physics is the presence, in the phase diagram of many physical systems, of “critical points” where some thermodynamic quantities or their derivatives diverge. Experimentally, it is found that such divergences are usually of power law type, and therefore characterized by exponents, called critical exponents. The theory of critical phenomena, developed to explain this behavior, suggests the existence of large “universality classes” such that systems belonging to the same universality class have the same critical exponents.

A closely related notion of universality concerns the “continuum scaling limit,” which is obtained by sending the *microscopic scale* of the system (e.g., the mesh of the lattice,

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for discrete systems defined on a lattice) to zero, while focusing on features manifested on a *macroscopic scale*. Such a limit is only meaningful at the critical point, where the *correlation length* (i.e., the “natural” length scale that characterizes the system) is supposed to diverge.

The concept of universality and the existence of universality classes arise naturally in the theory of critical phenomena (based on Renormalization Group techniques), and are backed by strong theoretical and experimental evidence. Nonetheless, very few rigorous results are available, especially below the upper critical dimension, where the values of the critical exponents are expected to be different from those predicted by mean-field theory (there are, however, some exceptions – see, e.g., [45, 46, 40, 15, 16, 11, 9, 36, 19, 7] for results concerning percolation and the Ising model). The main aim of this paper is to present some rigorous results in support of the idea of universality in the context of percolation theory in two dimensions (see [29, 26] for detailed accounts on percolation theory).

In recent years, substantial progress has been made in understanding two-dimensional critical percolation and its conformally invariant scaling limit (see, e.g., [42, 43, 39, 20, 12, 21, 13, 14], and in computing the percolation critical exponents by means of mathematically rigorous methods [44, 30]. The new tool that made this possible is the Stochastic Loewner Evolution introduced by Schramm [38], together with Smirnov’s proof [42, 43] of the conformal invariance of the scaling limit of crossing probabilities for site percolation on the triangular lattice.

Here we will focus on a class of percolation models, called enhancement percolation, introduced in [2] (see also [26]). Enhancement percolation configurations are obtained by modifying, according to some local set of rules, the configurations generated by an independent (Bernoulli) percolation process. Other dependent percolation models obtained in a similar way arise naturally in different contexts such as the nonequilibrium dynamics of stochastic Ising models at zero temperature (see, for instance, [8, 25, 27, 34, 35, 10, 23]) and cellular automata like bootstrap percolation (see, for instance, [3, 37, 22]).

The enhancements that we are interested in are endowed with a natural monotonicity property and are such that they do not change the nature of the phase transition and do not shift the critical point p_c , therefore transforming an independent critical percolation model into a different, but still critical, dependent percolation model. However, even an enhancement that does not shift p_c can modify significantly the initial percolation process, and it could a priori induce “macroscopic” changes; it is therefore natural to ask whether such an enhancement can change the continuum scaling limit and the critical exponents. We show that this is not the case for the class of enhancements studied here.

The result is not surprising, if seen in the context of the Renormalization Group picture which suggests that, as long as the enhancement has finite range, it should not affect the scaling limit and the critical exponents (see, for example, [18]). However, the Renormalization Group picture remains largely nonrigorous and no general formalism has been developed so far to put it on more solid ground. Moreover, as the eight-vertex model [5] and some spin models [31, 32, 33] show, the concept of universality needs to be taken with some care.

Universality results in the same spirit as those presented in this paper can be found in [15, 11, 16, 9] (in [28], universality for critical exponents is tested numerically). Generally speaking, one way to look at these results is as an attempt to test the “robustness” of the scaling limit and critical exponents, as well as to give examples of a strong form of universality by constructing different (dependent) percolation models with the same critical exponents and scaling limit as independent percolation. The percolation models considered in [15, 11, 16] (and in [28]) are generated by applying certain cellular automata to independent percolation configurations. In those cases, the initial percolation configurations and the rules of the cellular automata possess a global “spin-flip” symmetry. The situation in [9] and in this paper is different because the dynamics (or the enhancement) breaks the global “spin-flip” symmetry.

A key tool in proving the universality results is a coupling between the independent percolation process (before the enhancement) and the enhanced one, which allows to compare the two processes. This method does not apply directly to essential enhancements that do shift the critical point (because in that case, if one starts at the critical point, the enhanced process is supercritical). That situation is very interesting and deserves to be studied, but unfortunately the methods used in this paper do not seem to be useful there.

Before addressing the universality issue, we present a new result about two-dimensional enhancements endowed with a natural monotonicity property. This complements one of the main results of [2], where it is proved that certain enhancements, called essential, always shift the critical point p_c . We show that a two-dimensional, monotonic enhancement that is not essential cannot produce an infinite cluster, and therefore leaves the critical point unchanged. In particular, this means that the monotonic, nonessential enhancement of a critical (Bernoulli) percolation process is still critical, and that the phase transition in the enhanced model is still second order (or continuous). This motivates the rest of the paper, since it identifies a class of critical, dependent percolation models for which it is natural to ask about critical exponents and scaling limits.

The rest of the paper is organized as follows. The next section contains a preview of some of the main universality results, in the context of site percolation on the triangular lattice. Analogous universality results for the square and the hexagonal lattice are stated later on, in Section 4. Before that, enhancement percolation is precisely defined and discussed in Section 3, where new results about monotonic enhancements in two dimensions are also presented. The last two sections are dedicated respectively to the proofs of the results on enhancement percolation (Section 5) and of the universality results (Section 6).

2 Preview of the Universality Results

In this section, we present some of the main results of the paper. For a precise definition of enhancement percolation and more details, we refer to Section 3 below. The universality results collected in this section are limited for simplicity to the case of the triangular lattice, where they can be stated unconditionally since the existence of (certain) critical exponents, of the scaling limit of crossing probabilities, and of the full scaling limit have

been rigorously proved. Later, in Section 4, we will state some results for the square and the hexagonal lattice, which will be however conditional on the existence of the critical exponents and the scaling limit for independent (site) percolation on those lattices.

2.1 Monotonic Enhancements

Consider a dependent (site) percolation model on a regular lattice \mathbb{L} in which an initial configuration is generated by independent variables having density p and then enhanced by means of a local function of the configuration. By regular lattices we mean the class of infinite graphs considered in [29], but except for Appendix A, for simplicity we will restrict our attention to the square, triangular and hexagonal lattice. The enhancement is stochastically activated at each site with probability s , independently of the other sites, and its effect is to (possibly) make certain closed sites open. Let $\theta(p, s)$ be the percolation probability of the enhanced process, i.e., the probability that the origin belongs to an infinite open cluster after the enhancement.

In this paper, we restrict our attention to a class of enhancements endowed with a natural *monotonicity* property, i.e., we consider enhancement functions that are nondecreasing in the number of open sites (see Section 3.2 for a precise definition).

While we postpone the precise definitions to Section 3.2, we give here a simple example. On a lattice \mathbb{L} of degree D , consider an enhancement that, when activated at the origin, makes it open if at least $m \leq D$ of its neighbors are open. This can be considered the prototypical example of the type of finite-range, monotonic enhancements that we are interested in. All enhancements in this paper have finite-range in the sense that, as in the example above, the effect of the enhancement depends only on a bounded subset of the percolation configuration around the location where the enhancement is activated. The enhancement in the example is monotonic in the sense that making more sites open in the original percolation configuration can only produce more open sites, and never inhibits the enhancement.

We remark that the restriction to monotonic enhancements is very natural, since if the enhancement is not monotonic, in general the percolation probability $\theta(p, s)$ will not be monotonic in p , and there could be ambiguity over the correct definition of the critical point (i.e., there could be more than one critical point—as an example, consider an enhancement of site percolation on the square lattice such that a closed site is made open only if all of its neighbors are closed).

The following question motivates the rest of the paper. What happens to the phase transition when a monotonic enhancement is applied at p_c ? With regard to this question, there could a priori be three types of monotonic enhancements: (1) enhancements that make the percolation process supercritical and therefore shift the critical point, (2) enhancements that do not make the process supercritical but change the universality class of the percolation model, (3) enhancements that do not make the process supercritical and do not change the universality class of the model. In what follows, we will essentially show that in two dimensions the second class of enhancements is empty. In other words, all two-dimensional dependent percolation models generated by a monotonic enhancement

acting on independent (Bernoulli) percolation and which does not shift the critical point are in the same universality class as independent percolation.

In dimension two, we also show (see Theorem 5 in Section 3.2) that the only monotonic enhancements that can shift the critical point are those that Aizenman and Grimmett [2] call *essential* (see Section 3.2 for the definition).

2.2 Critical Exponents

One of our universality results concerns four critical exponents, namely the exponents β (related to the percolation probability), ν (related to the correlation length), η (related to the connectivity function) and γ (related to the mean cluster size). The existence of these exponents has been recently proved (see [44, 30] for the details), and their predicted values confirmed rigorously, for the case of independent site percolation on the triangular lattice \mathbb{T} . Such exponents are believed to be universal for independent percolation on regular lattices in the sense that their value should depend only on the number of dimensions and not on the structure of the lattice or on the nature of the percolation model (e.g., whether it is site or bond percolation); that type of universality has not yet been proved.

Consider an independent percolation model on a regular lattice \mathbb{L} with configurations chosen from a Bernoulli product measure P_p with density of open sites p (E_p will denote expectation with respect to P_p). Assume that \mathbb{L} is such that $0 < p_c < 1$. Let C be the open cluster containing the origin and $||C||$ its cardinality, then $\theta(p) = P_p(||C|| = \infty)$ is the **percolation probability**. Arguments from theoretical physics suggest that $\theta(p)$ behaves roughly like $(p - p_c)^\beta$ as p approaches p_c from above.

It is also believed that the **connectivity function**

$$\tau_p(x) = P_p(\text{the origin and } x \text{ belong to the same open cluster}) \quad (1)$$

behaves, for large Euclidean norm $|x|$, like $|x|^{-\eta}$ if $p = p_c$, and like $\exp(-|x|/\xi(p))$ if $0 < p < p_c$, for some $\xi(p)$ satisfying $\xi(p) \rightarrow \infty$ as $p \uparrow p_c$. The **correlation length** $\xi(p)$ is defined by

$$\xi(p)^{-1} = \lim_{|x| \rightarrow \infty} \left\{ -\frac{1}{|x|} \log \tau_p(x) \right\}. \quad (2)$$

$\xi(p)$ is believed to behave like $(p_c - p)^{-\nu}$ as $p \uparrow p_c$. The **mean cluster size** $\chi(p) = E_p ||C||$ is also believed to diverge with a power law behavior $(p_c - p)^{-\gamma}$ as $p \uparrow p_c$.

One possible way to state these conjectures is the following:

$$\lim_{p \downarrow p_c} \frac{\log \theta(p)}{\log(p - p_c)} = \beta, \quad (3)$$

$$\lim_{|x| \rightarrow \infty} \frac{\log \tau_{p_c}(x)}{\log |x|} = -\eta, \quad (4)$$

$$\lim_{p \uparrow p_c} \frac{\log \xi(p)}{\log(p_c - p)} = -\nu, \quad (5)$$

$$\lim_{p \uparrow p_c} \frac{\log \chi(p)}{\log(p - p_c)} = -\gamma. \quad (6)$$

We call $\theta(p, s)$, $\xi(p, s)$, $\tau_{p,s}(x)$ and $\chi(p, s)$ the quantities analogous respectively to $\theta(p)$, $\xi(p)$, $\tau_p(x)$ and $\chi(p)$ for the enhanced process (with density of enhancement s). In the case of site percolation on the triangular lattice, we have the following result.

Theorem 1. *For every monotonic enhancement (of independent site percolation) on the triangular lattice that does not shift the critical point, the critical exponents β , η , ν and δ exist for the enhanced percolation process and have the same numerical values as for the independent process (before the enhancement).*

In Section 4.1 below, we will state a similar result for the square and the hexagonal lattice (Theorem 6), conditional on the existence of the critical exponents for independent (site) percolation on those lattices.

2.3 Cardy's Formula and the Full Scaling Limit

Consider the rescaled triangular lattice $\delta\mathbb{T}$. As the lattice spacing δ goes to zero, the limit of the probability of an open crossing in an arbitrary domain between two (distinct) selected portions of its boundary has been shown [42] to exist and to be a conformal invariant of the domain and the two portions of boundary. This allows to obtain a formula [17] for crossing probabilities, first derived by Cardy using nonrigorous methods and bearing his name.

We will show that a monotonic nonessential enhancement does not change the scaling limit of crossing probabilities. In particular, this implies the following result.

Theorem 2. *For every monotonic enhancement (of independent site percolation) on the triangular lattice that does not shift the critical point, the crossing probabilities of the enhanced process converge in the scaling limit to Cardy's formula.*

An analogous results for site percolation on the square and hexagonal lattices, but conditional on the existence of the scaling limit of crossing probabilities there, is given in Section 4.2 (see Theorem 7).

Crossing probabilities only give partial information on a percolation model and on its scaling limit $\delta \rightarrow 0$. One way to go beyond crossing probabilities is by considering the law of the random interfaces, along the edges of the dual lattice $\delta\mathbb{H}$, between open and closed clusters, as suggested by Schramm [38] (see Figure 3 in Section 4.3). The existence of the scaling limit of the collection of all interfaces and some of its properties have been derived in [13], where the limiting object is called the percolation *full* scaling limit.

We will show that a monotonic nonessential enhancement does not change the full scaling limit. In particular, this implies the following result.

Theorem 3. *For every monotonic enhancement (of independent site percolation) on the triangular lattice that does not shift the critical point, the full scaling limit of the enhanced process is the same as the full scaling limit of the independent process.*

3 Enhancement Percolation

Consider a dependent percolation process in which the initial configuration, generated by independent variables having density p , is enhanced by means of a local function of the configuration. In [2], Aizenman and Grimmett ask under which circumstances the new critical density differs in value from the critical density p_c of the initial independent percolation. Looking at site percolation on the d -dimensional cubic lattice as a prototypical example, they introduce a general approach to answer that question, and give a sufficient condition for the enhancement to be capable of shifting the critical point. An analogous question is relevant for all models, such as Ising ferromagnets and the contact process, that are endowed with certain monotonicity properties with respect to the critical point (e.g., the addition of ferromagnetic couplings can only increase the transition temperature).

Loosely speaking, an enhancement is a systematic addition of open sites performed by means of a translation-invariant procedure with local rules; if it is capable of creating a percolation “backbone,” i.e., a doubly-infinite open path, then it is called *essential*. A main result of [2] is that, if $0 < p_c < 1$, an essential enhancement always shifts the critical point.

Clearly, not all enhancements are essential. As an example of a nonessential enhancement consider the addition of an open site at x with probability $\frac{1}{2}p_c$ whenever all the neighbors of x are closed. Such an enhancement introduces new open sites, but it cannot produce a doubly-infinite open path (almost surely).

3.1 The Lattices and Some Notation

We set up here the notation needed in the following sections. We will state most of our results for three planar lattices, the square, triangular and hexagonal lattice. The triangular and hexagonal lattices will be denoted by \mathbb{T} and \mathbb{H} , respectively. In Section 4.3, however, we will restrict our attention to site percolation on the triangular lattice only. Rather than treating the three lattices separately, we provide a unified treatment which in fact allows for even greater generality.

Remark 3.1. *We note that the results of Sections 3.2, 3.3 and 4.1 apply to general regular lattices of the type considered by Kesten in [29] (see Chapter 2 of [29] and Appendix A).*

Let \mathbb{L} be either the square, triangular or hexagonal lattice, embedded in \mathbb{R}^2 as in Figures 1 and 2. We think of a lattice as a geometric object made of sites and edges, and denote by $V(\mathbb{L})$ the set sites of \mathbb{L} . If F a face of \mathbb{L} , we call the **perimeter** of F the set of edges delimiting F , and denote by $V(F)$ the set of sites of \mathbb{L} along the perimeter of F .

Close-packing a face F of \mathbb{L} means adding an edge between each pair of vertices of F that do not already share an edge. In close-packing a face F , we shall choose to draw the new edges inside F , as in Figure 1. The lattice \mathbb{L}^* (the **matching graph** or **close-packed** version of \mathbb{L}) is obtained from \mathbb{L} by close-packing all its faces. Note that in the case of the triangular lattice (or any “triangulated” lattice), the close-packed version of the lattice coincides with the original lattice. Such a lattice is called **self-matching**.

We also introduce the **dual lattice** \mathbb{L}_d , whose sites (called **dual sites**) are the (centers of the) faces of \mathbb{L} . Two dual sites are neighbors when the perimeters of the corresponding faces have a common edge. We embed \mathbb{L}_d in \mathbb{R}^2 in such a way that each **dual edge** crosses an edge of \mathbb{L} (see Figure 2), and denote by $e_{x,y}^d = (x,y)_d$ the edge dual to the edge $e_{x,y} = (x,y)$ of \mathbb{L} (note that the sites that appear in this notation are not the dual sites on which the edge is incident). It is easy to see (Figure 2) that there is a duality relation between the triangular and the hexagonal lattice. We will use it in dealing with the full scaling limit of critical site percolation on \mathbb{T} .

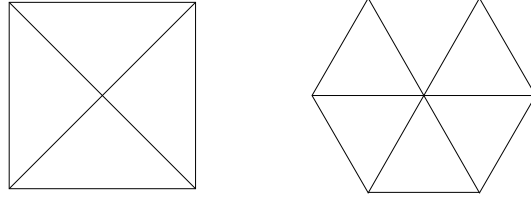


Figure 1: The close-packing of the elementary cells of the square and of the hexagonal lattice.

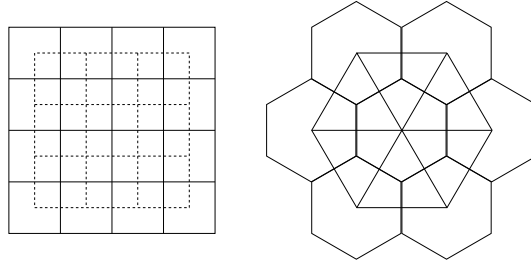


Figure 2: As shown on the left, the square lattice is self-dual. The duality between the triangular and the hexagonal lattice is shown on the right.

Two sites that are neighbors in \mathbb{L} (respectively, \mathbb{L}^* or \mathbb{L}_d) will also be called **\mathbb{L} -adjacent** (resp., **\mathbb{L}^* -** or **\mathbb{L}_d -adjacent**). Similarly, two subsets of \mathbb{L} are said to be **\mathbb{L} -adjacent** (resp., **\mathbb{L}^* -** or **\mathbb{L}_d -adjacent**) if the first one contains at least one site that is \mathbb{L} -adjacent (resp., \mathbb{L}^* - or \mathbb{L}_d -adjacent) to a site of the second one.

An **\mathbb{L} -path** (resp., **$*$ -path** or **dual path**) will be an alternating sequence of \mathbb{L} -adjacent (resp., \mathbb{L}^* - or \mathbb{L}_d -adjacent) sites and the edges between them. The set of sites of a path γ will be denoted by $V(\gamma)$. If the path is closed, in the sense that the initial and final sites coincide, it will be called a **loop**. Sometimes we will simply use the term path, without any specification, if there is no risk of confusion. A set $C \in V(\mathbb{L})$ is **\mathbb{L} -connected** (resp., **$*$ -connected**) if $\forall x, y \in C$, there exists an \mathbb{L} -path (resp., $*$ -path) from x to y that uses only sites in C .

Sites of \mathbb{L} and \mathbb{L}^* will be denoted by the Latin letters x , y and z , with the origin denoted by o , and paths on the two lattices by γ or λ and γ^* or λ^* , respectively. Dual sites will be denoted by the Greek letter ξ .

A self-avoiding loop J (i.e., a loop that does not have self-intersections) is a Jordan curve. Therefore, by the Jordan theorem, $\mathbb{R}^2 \setminus J$ consists of a bounded component, denoted by $\text{int}(J)$, and an unbounded component, denoted by $\text{ext}(J)$. Notice that, in the case of a planar lattice, any site self-avoiding path is self-avoiding. A standard loop-removal procedure to extract from a generic path a site self-avoiding subpath with the same initial and final sites is described in Chapter 2 of [29].

The **external (site) boundary** of a set $C \in V(\mathbb{L})$ is the set of sites of \mathbb{L} that are not in C but have at least one neighbor in C . It is a key observation (see, for example, [41] and Corollary 2.2 of [29]) that the external (site) boundary of a nonempty, bounded, \mathbb{L} -connected set C of sites of \mathbb{L} forms, together with the edges between sites in the boundary, a self-avoiding $*$ -loop λ^* such that all the sites in C belong to $\text{int}(\lambda^*)$. Note also that any nonempty, bounded, \mathbb{L} -connected set of sites of \mathbb{L} is surrounded by a self-avoiding dual loop.

3.2 Essential and Monotonic Enhancements

Although the choice of bond or site percolation is irrelevant for our purposes, we will consider, for definiteness, a site percolation model on \mathbb{L} with configuration $\eta \in \{0, 1\}^{V(\mathbb{L})}$ chosen from a Bernoulli product measure P_p with density of open sites p (E_p will denote expectation with respect to P_p). One reason for dealing with site percolation is that every bond model can be reformulated as a site model on a different lattice (the converse is not true and therefore site models are more general than bond models – for more details, see [29]). We interpret the value $\eta(x) = 1$ as meaning that $x \in V(\mathbb{L})$ is open and $\eta(x) = 0$ as meaning that it is closed, and represent each realization of the process by the collection $\omega = \{x \in V(\mathbb{L}) : \eta(x) = 1\}$ of open sites. We will often call ω a configuration, making no distinction between ω and η , and will denote by Ω the set of all ω 's. We call a path **open** (resp., **closed**) if all sites on the path are open (resp., closed). We say that an open path of η is contained in ω since all its sites are in ω .

Let $B(r) = \{u \in \mathbb{R}^2 : |u| \leq r\}$, where $|\cdot|$ is the Euclidean norm. Following [2], we call $\phi_o : \Omega \rightarrow \Omega$ an **enhancement function** if, for each ω , it satisfies the following (locality) properties:

- $\phi_o(\omega)$ depends only on the restriction of ω to $\mathbb{B}_o = V(\mathbb{L}) \cap B(R_0)$, for some fixed $R_0 < \infty$,
- $\phi_o(\omega) \subset \mathbb{B}_o$.

R_0 is called **enhancement range**. We extend ϕ_o by translations to a family $\phi = \{\phi_x : x \in V(\mathbb{L})\}$ of functions associated with the lattice sites: $\phi_x(\omega) = x + \phi_o(\tau_x \omega)$, where τ_x is the shift operator on Ω given by $\tau_x \omega(y) = \omega(y + x)$. We shall also consider a collection $\alpha = \{\alpha(x) : x \in V(\mathbb{L})\}$ of i.i.d. random variables independent of η , taking values in

$\{0, 1\}$, and interpret the value $\alpha(x) = 1$ as meaning that the enhancement at site x is “activated.” We denote by $s = P'_s(\alpha(x) = 1)$ the **density of enhancement**, where P'_s is a Bernoulli product measure independent of P_p . The **enhanced configuration** $\hat{\omega} = \hat{\omega}(\omega, \alpha)$ is then defined as

$$\hat{\omega} = \omega \cup \left(\bigcup_{x: \alpha(x)=1} \phi_x(\omega) \right) \quad (7)$$

and the corresponding $\hat{\eta}$ is obtained by declaring open, regardless of their state in η , all the sites contained in $\phi_x(\omega)$ for each x such that $\alpha(x) = 1$.

Let us now make the concept of essential enhancement precise. We say that a path is **self-repelling** (see p. 66 of [26]) if none of its sites is adjacent to any other site of the path except for its two neighbors in the path (one neighbor, in the special case of the first and last site of the path). An enhancement is called **essential** if there exists a configuration ω containing no doubly-infinite, self-repelling path, but such that the enhancement at the origin produces such a path.

Notice that for every path γ between x and y , there always exists at least one self-repelling subpath between the same sites. To see this, take a shortest (in terms of number of sites) subpath γ' of the original path that connects x and y . If that were not a self-repelling path, it would contain at least two adjacent sites which are not neighbors in the path. By deleting from γ' all the sites and edges between those two sites (in the order inherited from the original path), one would produce a subpath of γ that is shorter than γ' , thus getting a contradiction.

Like for a path, we say that a loop is self-repelling if none of its sites is adjacent to any other site of the loop, except for its two neighbors in the loop. If the interior of a loop λ is not empty and $x \in \text{int}(\lambda)$, then there exists at least one self-repelling subloop whose interior also contains x . The proof of this fact proceeds just like the one outlined above for a path.

We denote by $\theta(p)$ the probability that the origin belongs to an infinite open cluster in the original process, and by $\theta(p, s)$ the corresponding probability in the (stochastically) enhanced process. Clearly, $\theta(p, 0) = \theta(p)$ and $\theta(p, s)$ is a monotonic function of s . Notice however that, while $\theta(p)$ is a monotonic function of p , this is not necessarily true of $\theta(p, s)$. The monotonicity of $\theta(p, s)$ in p depends on the nature of the enhancement function ϕ_o . Let \leq denote the natural partial order on the set $\{0, 1\}^{V(\mathbb{L})}$, which corresponds to the partial order induced on the set Ω by the notion of inclusion. We call the enhancement function ϕ_o **monotonic** (see p. 64 of [26]) if, for all $\eta \leq \eta'$ (or, equivalently, $\omega \subset \omega'$), $\phi_o(\omega)$ is a subset of $\phi_o(\omega')$. An enhancement defined by a monotonic enhancement function is itself called monotonic.

Denoting by p_c the critical probability of independent site percolation, Aizenman and Grimmett [2] prove the following property of essential enhancements (although for definiteness they restrict attention to the case of site percolation on the d -dimensional cubic lattice, with dimension $d \geq 2$, their method is more general and applies to other lattices as well).

Theorem 4. [2] *Suppose $p_c > 0$, and let $s > 0$. For any essential enhancement, there exists a nonempty interval $(\pi(s), p_c)$ such that $\theta(p, s) > 0$ when $\pi(s) < p < p_c$.*

The reader should be warned against the temptation to weaken the condition that the enhancement be essential. The following are two examples taken from [2] of slightly weaker conditions, neither of which is sufficient to guarantee Theorem 4 (see [2] for an example of enhancement satisfying 1 and 2, but for which Theorem 4 does not hold).

1. There exists a configuration ω which contains no infinite cluster, but for which there is an α such that the enhanced configuration $\hat{\omega}(\omega, \alpha)$ contains an infinite cluster.
2. There exists an enhanced configuration $\hat{\omega}(\omega, \alpha)$ which contains no doubly-infinite self-repelling path if $\alpha(0) = 0$, but contains such a path if $\alpha(0) = 1$.

An essential enhancement clearly satisfies condition 2, but the converse is not generally true. However, it is easy to see that a monotonic enhancement ϕ that satisfies condition 2 is essential by considering the configuration $\omega' = \hat{\omega}(\omega, \alpha)$ with $\alpha(0) = 0$. By condition 2, ω' does not contain a doubly-infinite self-repelling path, but since ϕ is monotonic and $\omega \subset \omega'$, if we start with ω' and activate the enhancement only at the origin, a doubly-infinite self-repelling path is produced. Therefore, we have constructed a configuration ω' without a doubly-infinite self-repelling path, but such that applying the enhancement at the origin produces such a path, which shows that ϕ is essential.

It is an immediate consequence of Theorem 4 that an essential enhancement satisfies condition 1. The next lemma states that for monotonic enhancements *in two dimensions* the converse is also true.

Lemma 3.1. *Let ϕ be a monotonic enhancement in two dimensions. ϕ is essential if and only if it satisfies condition 1.*

Lemma 3.1 implies that a monotonic enhancement that is not essential cannot create an infinite cluster; therefore, for all $p \leq p_c$, with probability 1 there is no infinite cluster in the enhanced percolation process. We postpone the proof of Lemma 3.1 to Section 5, but point out that it immediately yields the following result.

Corollary 3.1. *A monotonic nonessential enhancement in two dimensions does not change the nature of the phase transition – i.e., second order or continuous – and leaves the critical point unchanged.*

Corollary 3.1 implies that the monotonic nonessential enhancement of a critical percolation process is still critical, identifying a class of critical, dependent percolation models for which it is natural to ask about critical exponents and scaling limits. The restriction to monotonic enhancements is very natural, as remarked in Section 2.1. In fact, if ϕ_o is not monotonic, then in general $\theta(p, s)$ is not monotonic in p , which can create ambiguity over the correct definition of the critical point (i.e., there can be more than one critical point). For example, it is easy to think of nonmonotonic, nonessential enhancements that can produce an infinite cluster when the density of the original percolation process is close

to zero, but fail to do so when p is just below p_c (consider, for instance, site percolation on the square lattice, and an enhancement that makes a site open only if all of its neighbors are closed).

Corollary 3.1 and Theorem 4 combined imply the following result.

Theorem 5. *Suppose that $p_c > 0$, and let $s > 0$. For any monotonic enhancement in two dimensions, there exists $\pi(s) < p_c$ such that $\theta(p, s) > 0$ when $p > \pi(s)$ if and only if the enhancement is essential.*

Remark 3.2. *In view of Theorem 5, in the theorems of Section 2, one can substitute “For every monotonic enhancement that does not shift the critical point” with “For every monotonic nonessential enhancement.”*

3.3 A Useful Trick

We next show how to construct from an enhancement function ϕ_o a new local function $\Phi_o : \Omega \rightarrow \{o, \emptyset\}$ and a special enhanced configuration which will be useful in the proofs of some of the results. Φ_o is defined as follows:

$$\Phi_o(\omega) = \begin{cases} \{o\} & \text{if } \{o\} \subset \phi_x(\omega) \text{ for some } x \in \mathbb{B}_o \\ \{\emptyset\} & \text{otherwise} \end{cases} \quad (8)$$

Notice that Φ_o is an enhancement function with enhancement range $R = 2R_o$. As we did previously with ϕ_o , we extend Φ_o by translations to a family $\Phi = \{\Phi_x : x \in V(\mathbb{L})\}$ of functions associated with the lattice sites.

In general, Φ_o is different from ϕ_o , but it is monotonic whenever ϕ_o is, and has the following useful property.

Lemma 3.2. *The deterministic enhancement with density $s = 1$ defined by Φ is the same as the deterministic enhancement with density $s = 1$ defined by ϕ .*

Proof. It is enough to observe that $\bigcup_{x \in V(\mathbb{L})} \Phi_x(\omega) = \bigcup_{x \in V(\mathbb{L})} \phi_x(\omega)$. \square

The configuration obtained by a deterministic enhancement with density $s = 1$ will play an important role later, in some of the proofs; we will denote it by $\tilde{\omega}$ (and $\tilde{\eta}$).

The function Φ_o is in general simpler than ϕ_o , since it takes values in $\{o, \emptyset\}$ and its effect on η is to add at most one open site at the origin. For Φ_o , being essential means that there exists a configuration ω that does not contain the origin and does not contain a doubly-infinite self-repelling path, and such that $\Phi_o(\omega) = \{o\}$ and $\omega \cup \{o\}$ contains a doubly-infinite self-repelling path (which necessarily contains the origin). The next lemma shows that, if ϕ_o is monotonic, in order to decide whether it is essential or not, we may as well consider Φ_o .

Lemma 3.3. *Let ϕ_o be a monotonic enhancement function. ϕ_o is essential if and only if Φ_o is essential.*

Proof. The fact that if Φ_o is essential then ϕ_o must also be essential is true even if ϕ_o is not monotonic. To see it, take a configuration ω that does not contain the origin and does not contain a doubly-infinite self-repelling path, and such that $\Phi_o(\omega) = o$ and $\omega \cup o$ contains a doubly-infinite self-repelling path. $\Phi_o(\omega) = o$ means that $o \subset \phi_x(\omega)$ for some $x \in \mathbb{B}_o$. Then $\tau_x \omega$ is a configuration without a doubly-infinite self-repelling path, but such that applying the enhancement defined by ϕ_o at the origin produces such a path, which shows that ϕ_o is essential.

To prove the other direction of the claim, assume that ϕ_o is monotonic and essential and consider a configuration ω which does not contain a doubly-infinite self-repelling path, but such that $\omega \cup \phi_o(\omega)$ contains at least one such path. Let S be the set of sites contained in $\phi_o(\omega)$ that belong to the doubly-infinite self-repelling path(s) of $\omega \cup \phi_o(\omega)$; we enumerate them in some deterministic way and denote them by $S = \{x_1, \dots, x_k\}$. We now define a new configuration $\omega' = \omega \cup \left(\bigcup_{i=1}^{k'-1} \{x_i\} \right)$ as the unique configuration that does not contain a doubly-infinite self-repelling path but such that $\omega' \cup \{x_{k'}\}$ contains such a path. The monotonicity of ϕ_o implies that $\{x_{k'}\} \subset \phi_o(\omega')$, from which it is easy to see that $\Phi_o(\tau_{x_{k'}} \omega') = \{o\}$ and $\tau_{x_{k'}} \omega' \cup \{o\}$ contains a doubly-infinite self-repelling path, which shows that Φ_o is essential. \square

3.4 Two Simple Examples

We give here two simple examples of monotonic nonessential enhancements, to show that they do exist. 1) On the square lattice, consider an enhancement that when activated at the origin makes it open if its neighbors to the “north,” “east” and “west” are open. 2) On the triangular lattice, consider an enhancement that when activated at the origin makes it open if at least m of its neighbors are open.

The first enhancement is essential for site percolation on the square lattice, as can be easily seen by taking a configuration whose only open sites are the “north,” “east” and “west” neighbors of the origin and the sites in two non-adjacent, self-repelling paths starting from the neighbors to the “east” and to the “west” of the origin. Activating the enhancement at the origin would join the two paths into a doubly-infinite open path. However, the same enhancement is nonessential for $*$ -percolation (i.e., site percolation on the close-packed version of the square lattice) since the “north,” “east” and “west” neighbors of the origin already form a $*$ -connected set, so that making the origin open does not “enhance” the connectivity. By a similar reasoning, one can easily see that the second enhancement is essential if $m \leq 4$, but nonessential if $m = 5$ or 6 .

As the two examples show, whether an enhancement is essential or not depends on the geometry of the lattice. Essential enhancements are able to target special locations where the addition of an open site has a significant effect on the connectivity of the clusters.

4 More Universality Results

In the following sections, we will consider three different (but closely related) aspects of universality. Two of them concern the continuum scaling limit, which is obtained by considering the percolation model realized on the lattice $\delta\mathbb{L}$ and letting the mesh δ of the rescaled lattice go to 0. In the scaling limit, the range of the enhancement also gets scaled by a factor δ (i.e., the enhancement range becomes δR_0).

We note that, although the results of the next sections are stated for stochastically activated enhancements, it will be clear from the proofs that they are also valid for deterministically activated enhancements.

4.1 Critical Exponents

Consider independent (site) percolation and enhancement percolation on \mathbb{L} . (We remind the reader that \mathbb{L} is either the square, triangular, or hexagonal lattice.) The following holds.

Lemma 4.1. *For every monotonic nonessential enhancement, there exist constants $0 < c_1, c_2 < \infty$ such that, $\forall s \in [0, 1]$ and $|x|$ large enough,*

$$\theta(p) \leq \theta(p, s) \leq c_1 \theta(p) \quad \text{for } p \in (p_c, 1], \quad (9)$$

$$\tau_p(x) \leq \tau_{p,s}(x) \leq p^{-c_2} \tau_p(x), \quad \text{for } p \in (0, p_c], \quad (10)$$

$$\xi(p, s) = \xi(p), \quad \text{for } p \in (0, p_c]. \quad (11)$$

Theorem 1 of Section 2, as well as the next result, Theorem 6, follow immediately from Lemma 4.1.

Theorem 6. *Suppose that the critical exponents β , η , ν and δ exist for independent site percolation on \mathbb{L} . Then, for every monotonic nonessential enhancement, the critical exponents β , η , ν and δ exist also for the enhanced percolation process and have the same numerical values as for the independent process (before the enhancement).*

The proofs of Lemma 4.1 and Theorem 6, as well as of Theorem 1 of Section 2.2, are given in Section 6.1.

4.2 Crossing Probabilities

The fact that we embedded the square, triangular and hexagonal lattices as regular tessellations of the plane (see Figures 1 and 2) means that they can be partitioned into equal “cells,” a property that will be used in the proof of Theorem 7 below.

We look at the percolation model $\hat{\eta}$ on $\delta\mathbb{L}$ and consider the scaling limit, as $\delta \rightarrow 0$, of crossing probabilities, focusing for simplicity on the probability of an open crossing of a rectangle aligned with the Cartesian coordinate axes. A similar approach would work for any domain with a “regular” boundary, but it would imply dealing with more complex deformations of the boundary than that needed for proving the result for a rectangle.

Consider a finite rectangle $\mathcal{R} = \mathcal{R}(b, h) = (-b/2, b/2) \times (-h/2, h/2) \subset \mathbb{R}^2$ centered at the origin of \mathbb{L} , with sides of lengths b and h and aspect ratio $\rho = b/h$. We say that there is an open vertical \mathbb{L} -crossing of \mathcal{R} in η (respectively, $\hat{\eta}$) if $\mathcal{R} \cap \delta\mathbb{L}$ contains an \mathbb{L} -path of open sites from η (resp., $\hat{\eta}$) joining the top and bottom sides of the rectangle \mathcal{R} , and call $\varphi_\delta(b, h)$ (resp., $\hat{\varphi}_\delta(b, h)$) the probability of a such an open crossing. More precisely, there is a vertical open crossing in η (resp., $\hat{\eta}$) if there is an \mathbb{L} -path $(x_0, e_{x_0, x_1}, x_1, \dots, x_m, e_{x_m, x_{m+1}}, x_{m+1})$ in \mathbb{L} with $\underline{\eta(x_j)} = 1$ (resp., $\underline{\hat{\eta}(x_j)} = 1$) for all j , with $\delta x_1, \dots, \delta x_m$ all in \mathcal{R} , and with the line segments $\delta x_0, \delta x_1$ and $\delta x_m, \delta x_{m+1}$ touching respectively the top side $[-b/2, b/2] \times \{h/2\}$ and the bottom side $[-b/2, b/2] \times \{-h/2\}$.

It is believed that the scaling limit of crossing probabilities for independent critical percolation in two dimensions exists, is universal, and is given by Cardy's formula [17, 18]. However Cardy's formula has so far been rigorously proved only in the case of critical site percolation on the triangular lattice [42, 43], for which we have already presented a result in Section 2.3 (see Theorem 2 there). For the case of the square and hexagonal lattice, assuming that $\lim_{\delta \rightarrow 0} \varphi_\delta(b, h) = F(\rho)$, where F is a continuous function, we have the following result.

Theorem 7. *Let \mathbb{L} be the square or hexagonal lattice, and assume that the scaling limit of crossing probabilities of a rectangle \mathcal{R} exists for independent critical site percolation on \mathbb{L} and is given by a continuous function F of ρ . Then, for every monotonic nonessential enhancement, the corresponding crossing probabilities in the enhanced process have the same scaling limit F .*

The proof of Theorem 7 (and of Theorem 2 of Section 2.3) is given in Section 6.2.

4.3 The Full Scaling Limit

In this section we further restrict attention to the triangular lattice \mathbb{T} , whose sites we think of as the (centers of the) elementary cells of a regular hexagonal lattice \mathbb{H} embedded in the plane as in Figure 3. In this case, $\mathbb{L} = \mathbb{L}^* = \mathbb{T}$ and there is no difference between \mathbb{L} -paths and $*$ -paths.

The full scaling limit of two-dimensional critical percolation was described in [12]; it represents the limit as $\delta \rightarrow 0$ of the collection of all the boundaries between open and closed clusters at $p = p_c$. Its existence and some of its properties have been proved in [13] for the case of critical site percolation on \mathbb{T} . In dealing with the scaling limit, as in [12, 13], we adopt the Aizenman-Burchard approach [1]. A precise formulation requires some additional notation, given below.

The edge $e_{x,y}^d = (x, y)_d$ of \mathbb{H} , dual to the edge (x, y) of \mathbb{T} is said to be **unsatisfied** in η (resp., in $\hat{\eta}$) if $\eta(x) \neq \eta(y)$ (resp., $\hat{\eta}(x) \neq \hat{\eta}(y)$). We call $\Gamma(\eta)$ (resp., $\Gamma(\hat{\eta})$) the set of unsatisfied dual edges in the configuration η (resp., $\hat{\eta}$). The dual edges in Γ make up the interfaces between open and closed clusters. More precisely, an open (resp., closed) cluster is a maximal connected subset of \mathbb{T} whose sites are all open (resp., closed), and we call **boundary** between an open and a closed cluster the collection of unsatisfied dual edges that lie between sites of the two clusters.

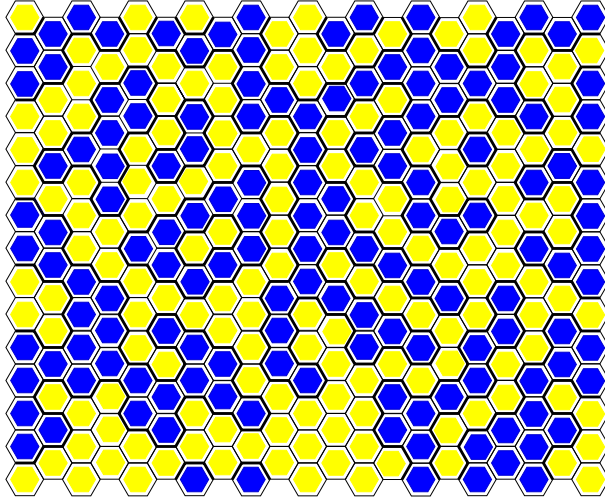


Figure 3: Finite portion of a (site) percolation configuration on the triangular lattice \mathbb{T} , where the sites of \mathbb{T} are represented as faces of the hexagonal lattice \mathbb{H} . The boundaries between clusters are indicated by heavy lines.

A **boundary path** (**b-path** for short) is an *oriented, self-avoiding* \mathbb{H} -path $\vec{\gamma}_d = \{\xi_0, e_1^d, \dots, e_{k-1}^d, \xi_k\}$ written as an ordered, alternating sequence of sites of \mathbb{H} and the edges between them, such that all edges in $\vec{\gamma}_d$ belong to Γ . If the path $\vec{\gamma}_d$ forms a loop (i.e., if $\xi_k = \xi_0$), it will be called a **b-loop**. In this case, $\vec{\gamma}_d$ coincides with a complete boundary. As boundaries between open and closed clusters, b-paths can always be extended to form a loop or a doubly-infinite path. When there is no infinite cluster, like in two-dimensional critical Bernoulli percolation, all complete boundaries are b-loops.

We denote by \mathcal{F}_δ a collection of b-loops of step size δ , which we identify with the boundaries of independent (Bernoulli) percolation on the lattice $\delta\mathbb{T}$ at $p = p_c (= 1/2)$, and by $\hat{\mathcal{F}}_\delta$ the collection of the boundaries obtained from \mathcal{F}_δ by enhancement. We note that independent percolation and the enhanced one are coupled on the probability space $(\{0, 1\}^{V(\mathbb{T})} \times \{0, 1\}^{V(\mathbb{T})}, \Sigma, \mathbb{P})$, where $\mathbb{P} = P_p \times P'_s$ and Σ is the σ -algebra generated by cylinder events. We call μ_δ the distribution of \mathcal{F}_δ and $\hat{\mu}_\delta$ the distribution of $\hat{\mathcal{F}}_\delta$. These collections of paths (or rather, their distributions), indexed by δ , are the objects of which we take the continuum scaling limit, letting $\delta \rightarrow 0$.

The scaling limit $\delta \rightarrow 0$ can be taken by focusing on fixed finite regions, $\Lambda \subset \mathbb{R}^2$, or by treating the whole \mathbb{R}^2 . The second option avoids technical issues that arise near the boundary of Λ . A convenient way of dealing with the whole \mathbb{R}^2 is to replace the Euclidean metric with a distance function $d(\cdot, \cdot)$ defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by

$$d(u, v) = \inf_f \int (1 + |f|^2)^{-1} dl, \quad (12)$$

where the infimum is over all smooth curves $f(l)$ joining u with v , parametrized by arclength l , and $|\cdot|$ denotes the Euclidean norm. This metric is equivalent to the Euclidean

metric in bounded regions, but it has the advantage of making \mathbb{R}^2 precompact. Adding a single point at infinity yields the compact space \mathbb{R}^2 which is isometric, via stereographic projection, to the two-dimensional sphere.

Denote by \mathcal{S} the complete, separable metric space of continuous curves in \mathbb{R}^2 with a distance $D(\cdot, \cdot)$ based on the metric defined by equation (12) as follows. Curves are regarded as equivalence classes of continuous functions $g(t)$, from the unit interval $[0, 1]$ to \mathbb{R}^2 , modulo monotonic reparametrizations. The distance D between two curves of \mathcal{S} , \mathcal{C}_1 and \mathcal{C}_2 , is defined by

$$D(\mathcal{C}_1, \mathcal{C}_2) = \inf_{f_1, f_2} \sup_{t \in [0, 1]} d(g_1(f_1(t)), g_2(f_2(t))), \quad (13)$$

where g_1 and g_2 are particular parametrizations of \mathcal{C}_1 and \mathcal{C}_2 , and the infimum is over the set of all monotone (increasing or decreasing) continuous functions from the unit interval onto itself. The distance between two closed sets of curves, \mathcal{F} and \mathcal{F}' , is defined by the induced Hausdorff metric as follows:

$$(\text{dist}(\mathcal{F}, \mathcal{F}') \leq \varepsilon) \Leftrightarrow (\forall \mathcal{C} \in \mathcal{F}, \exists \mathcal{C}' \in \mathcal{F}' \text{ with } D(\mathcal{C}, \mathcal{C}') \leq \varepsilon, \text{ and vice versa}). \quad (14)$$

With these definitions, we have the following result.

Lemma 4.2. *The distance between the collection of random curves \mathcal{F}_δ from independent percolation and the corresponding collection of curves $\hat{\mathcal{F}}_\delta$ from the enhanced percolation process goes to zero almost surely as $\delta \rightarrow 0$; i.e., for \mathbb{P} -almost every (η, α) ,*

$$\lim_{\delta \rightarrow 0} \text{dist}(\mathcal{F}_\delta, \hat{\mathcal{F}}_\delta) = 0. \quad (15)$$

The main application of Lemma 4.2 is Theorem 3 of Section 2.3. The proofs of Lemma 4.2 and Theorem 3 are given in Section 6.3.

5 Proofs of the Enhancement Percolation Results

First of all we remind the reader of the definition of self-repelling path (see p. 66 of [26]), then we prove Lemma 3.1 and Theorem 5 of Section 3.2. We will need three “geometric” lemmas, two of which are easy to establish. The third one is of the type that can be deemed “obvious” (at least if one focuses on a specific lattice), but is nonetheless quite tedious to prove. The proofs, in a general setting, of those three lemmas are given in Appendix A.

Definition 5.1. *We say that a path is **self-repelling** if none of its sites is adjacent to any other site of the path except for its two neighbors in the path (one neighbor, in the special case of the first and last site of the path).*

Proof of Lemma 3.1. As already remarked, one direction is an immediate consequence of Theorem 4, so we only need to prove the other direction. Let us consider an enhancement that satisfies condition 1 and let ω be a configuration which does not contain an

infinite cluster, but such that an infinite cluster exists in the fully enhanced configuration $\hat{\omega}(\omega, \alpha)$ for some α . If this is the case, then also the configuration $\tilde{\omega}$ defined at the end of Section 3.2 contains a fortiori an infinite cluster. We shall also assume, without loss of generality, that $\tilde{\eta}(o) = 1$ (i.e., $\tilde{\omega}$ contains the origin) and that the origin belongs to the infinite cluster of $\tilde{\omega}$; then $\tilde{\omega}$ contains an infinite path γ starting at the origin.

Since ω contains no infinite cluster, η does not contain an infinite open cluster and the origin must be surrounded by infinitely many closed $*$ -loops. Take one such $*$ -loop that does not intersect $B(2R)$ and construct from it a self-repelling $*$ -loop λ^* whose interior contains $B(2R)$ (where $R = 2R_0$ is the range of Φ_o). By Lemma A.2, stated and proved in Appendix A, each site of λ^* has at least one neighbor in $V(\mathbb{L}) \cap \text{int}(\lambda^*)$ and one in $V(\mathbb{L}) \cap \text{ext}(\lambda^*)$. Moreover, by Lemma A.3, whose statement and proof can be found in the same appendix, $\mathbb{L} \cap \text{int}(\lambda^*)$ is \mathbb{L} -connected.

Every \mathbb{L} -path starting at the origin and going to infinity intersects λ^* . Let $x_0 \in \gamma \cap \lambda^*$ be the first site of λ^* intersected by γ parametrized from the origin to infinity, and consider $\lambda^* \cap B_{x_0}(R)$, where $B_{x_0}(R)$ denotes the open ball of radius R centered at x_0 . Starting from x_0 , check the sites of λ^* in both increasing and decreasing order and, in both directions, mark all those sites encountered before exiting $B_{x_0}(R)$ for the first time. Call S the set of sites marked in this process and let $\mathbb{B} = V(\mathbb{L}) \cap B(R)$. S partitions $B_{x_0}(R) \setminus S$ into two disjoint subsets such that no site of the first is \mathbb{L} -adjacent to a site of the second (see, e.g., Lemma A.2 of [29]). In the same way, its translate $\tau_{x_0}S$ partitions $\mathbb{B} \setminus \tau_{x_0}S$ in two disjoint subsets, \mathbb{B}_1 and \mathbb{B}_2 , that are not \mathbb{L} -adjacent.

Consider the following configuration η''' and the corresponding ω''' .

$$\eta'''(x) = \begin{cases} 1 & \text{if } x \in \mathbb{B} \text{ and } x \notin \tau_{x_0}S \\ 0 & \text{if } x \in \mathbb{B} \text{ and } x \in \tau_{x_0}S \\ 0 & \text{if } x \notin \mathbb{B} \end{cases} \quad (16)$$

We now introduce a configuration $\omega' = \omega''' \cup \omega''$, where ω'' contains only two infinite, disjoint, self-repelling \mathbb{L} -paths, γ_1 and γ_2 , which are not \mathbb{L} -adjacent and do not intersect $\tau_{x_0}S$, and such that γ_1 starts from an \mathbb{L} -neighbor of the origin in \mathbb{B}_1 and does not intersect \mathbb{B}_2 and γ_2 starts from an \mathbb{L} -neighbor of the origin in \mathbb{B}_2 and does not intersect \mathbb{B}_1 . By definition, ω' contains no doubly-infinite, self-repelling path, but such a path is produced if the origin is added.

Now notice that $\eta(x_0) = 0$ and $\tilde{\eta}(x_0) = 1$, which means that $\Phi_{x_0}(\omega) = x_0$ or equivalently $\Phi_o(\tau_{x_0}\omega) = o$. Since ϕ_o is monotonic, Φ_o is also monotonic, and since $\eta(x + x_0) \leq \eta'(x) \forall x \in \mathbb{B}$, it follows that $\Phi_o(\omega') = o$. Therefore, activating the enhancement defined by Φ_o at the origin in the configuration ω' produces a doubly-infinite, self-repelling path. The enhancement defined by Φ_o is therefore essential. This implies, by Lemma 3.3, that also the enhancement defined by ϕ_o is essential. \square

Proof of Theorem 5. One direction is already contained in Theorem 4 of Section 2.1. The other direction follows from Corollary 3.1 of Section 3, which is a straightforward consequence of Lemma 3.1. \square

6 Proofs of the Universality Results

We assume throughout this section that we are dealing with a monotonic nonessential enhancement ϕ of range R_0 . Remember that $R = 2R_0$ is the range of the enhancement Φ obtained from ϕ , as explained at the end of Section 3.2, and that the enhancement range gets rescaled by a factor δ when the lattice under consideration is $\delta\mathbb{L}$. In the rest of the paper, $B_u(r)$ will indicate the open ball of radius r centered at u . We will also need the following definitions and lemmas.

Definition 6.1. A site $x \in V(\mathbb{L})$ is called **protected** if $\eta(x) = 0$ and in η there are at least two closed $*$ -paths, $\gamma_1^* = \{z_0^1, e_1^1, z_1^1, \dots, z_{k_1}^1\}$ and $\gamma_2^* = \{z_0^2, e_1^2, z_1^2, \dots, z_{k_2}^2\}$, such that (i) $z_0^1 = z_0^2 = x$, (ii) $z_{k_1}^1, z_{k_2}^2 \notin B_x(2R)$, (iii) except for x , γ_1^* and γ_2^* are disjoint and nonadjacent.

Definition 6.2. A dual edge $e_{xy}^d = (x, y)_d$ such that x is protected and $\eta(y) = 1$ is called **stable**.

Protected sites satisfy the conditions of the next lemma, which is a key technical result. Its proof is similar to that of Lemma 3.1; we spell it out for the sake of completeness. We remind the reader that $\tilde{\eta}$ is the configuration obtained by a deterministic enhancement with $s = 1$ (see Section 3.3).

Lemma 6.1. Set $\delta = 1$ for simplicity. If $\eta(x) = 0$ and in η there are at least two closed $*$ -paths, γ_1^* and γ_2^* , starting at x but otherwise disjoint and nonadjacent in \mathbb{L}^* , which exit $B_x(R)$, then $\tilde{\eta}(x) = 0$.

Proof. We will prove the result by contradiction. Extract from γ_1^* and γ_2^* two self-repelling $*$ -paths, $\gamma_1'^*$ and $\gamma_2'^*$, starting at x , and let $\gamma^* = \gamma_1'^* \cup \gamma_2'^*$. Since $\gamma_1'^*$ and $\gamma_2'^*$ are disjoint and nonadjacent in \mathbb{L}^* , except for the common starting point x , and each one of them is self-repelling, γ^* is also self-repelling. Moreover, like the set S in the proof of Lemma 3.1, γ^* partitions $(V(\mathbb{L}) \cap B_x(R)) \setminus V(\gamma^*)$ into two disjoint, nonadjacent, \mathbb{L} -connected sets D_1 and D_2 . From Lemma A.2, site x has at least one \mathbb{L} -neighbor in D_1 and one in D_2 .

Consider the following configuration η''' and the corresponding ω''' .

$$\eta'''(y) = \begin{cases} 1 & \text{if } y \in \mathbb{B} \text{ and } y \notin \tau_x \gamma^* \\ 0 & \text{if } y \in \mathbb{B} \text{ and } y \in \tau_x \gamma^* \\ 0 & \text{if } y \notin \mathbb{B} \end{cases} \quad (17)$$

where $\mathbb{B} = V(\mathbb{L}) \cap B(R)$. Notice that, by Lemmas A.2 and A.3, $\tau_x \gamma^*$ partitions $\mathbb{B} \setminus V(\tau_x \gamma^*)$ into two disjoint, nonadjacent, \mathbb{L} -connected sets \mathbb{B}_1 and \mathbb{B}_2 , and that the origin, contained in $\tau_x \gamma^*$, has at least one \mathbb{L} -neighbor in \mathbb{B}_1 and one in \mathbb{B}_2 .

We now introduce a new configuration $\omega' = \omega''' \cup \omega''$, where ω'' consists of only two infinite, disjoint, self-repelling \mathbb{L} -paths, γ_1 and γ_2 , that are nonadjacent in \mathbb{L} and do not intersect $\tau_x \gamma^*$, and such that γ_1 starts from an \mathbb{L} -neighbor of the origin in \mathbb{B}_1 and does not intersect \mathbb{B}_2 and γ_2 starts from an \mathbb{L} -neighbor of the origin in \mathbb{B}_2 and does not intersect

\mathbb{B}_1 . By construction, ω' contains no doubly-infinite, self-repelling path, but such a path is produced if the origin is added.

Suppose now that $\tilde{\eta}(x) = 1$. Since the enhancement that produced $\tilde{\eta}$ is assumed monotonic and, by construction, $\eta(x + x_0) \leq \eta'(x) \ \forall x \in \mathbb{B}$, $\tilde{\eta}(x) = 1$ implies that, when the enhancement is activated at the origin in ω' , the origin is added to ω' and a doubly-infinite, self-repelling path is produced. From this it follows that the enhancement is essential, giving a contradiction. \square

Lemma 6.1 will be used several times, starting with the proof of another key technical result, Lemma 6.2 below.

Lemma 6.2. *Set $\delta = 1$ for simplicity. If x and y are protected, $B_x(2R) \cap B_y(2R) = \emptyset$, and in η there is a closed $*$ -path γ^* from x to y , then in $\tilde{\eta}$ there is a closed $*$ -path γ'^* from x to y .*

Proof. To prove the lemma, we first construct a new closed $*$ -path γ'^* joining x with y , and then show that we can apply Lemma 6.1 to all its sites. First of all, extract from γ^* a self-repelling $*$ -path γ''^* . Part of the new path that we are going to construct will coincide with γ''^* ; we just need to define γ'^* inside $B_x(2R)$ and $B_y(2R)$.

Let us start with $B_y(2R)$. Remember that y is protected and let $\gamma_{y,1}^*$ and $\gamma_{y,2}^*$ be the $*$ -paths of Definition 6.1. Call y' the first site of γ''^* (counting from x to y in the natural order associated with γ''^*) that is \mathbb{L}^* -adjacent to a site of either $\gamma_{y,1}^*$ or $\gamma_{y,2}^*$ (such a site always exists, although it may coincide with an \mathbb{L}^* -neighbor of y), and assume, without loss of generality, that y' is \mathbb{L}^* -adjacent to $\gamma_{y,1}^*$. An analogous construction (but with the ordering of the sites in γ''^* reversed, from y to x) gives a site $x' \in B_x(2R)$, and again we assume, without loss of generality, that x' is \mathbb{L}^* -adjacent to $\gamma_{x,1}^*$.

The path γ'^* is then obtained by pasting together the portion of $\gamma_{x,1}^*$ between x and x' , the portion of γ''^* between x' and y' , and the portion of $\gamma_{y,1}^*$ between y' and y in such a way that the resulting path is self-repelling. The sites in the new path need to be reordered, which can be easily done starting from x and using the order that each piece inherits from the original path it comes from (or that order inverted, for the last piece).

It is now easy to see that $\tilde{\eta}(z) = 0$ for all $z \in \gamma'^*$ by an application of Lemma 6.1. In order to do that, we just need to check that, for each site $z \in \gamma'^*$, there are two closed $*$ -paths that start at z , exit $B_z(R)$, and are not \mathbb{L}^* -adjacent, except for their common starting point z . If $z \in \gamma'^*$, but $z \notin B_x(R), B_y(R)$, this is obvious. It suffices to take the two portions of γ'^* from z to x and from z to y until they exit $B_z(R)$.

If $z \in \gamma'^*$ and, say, $z \in B_y(R)$, we construct the two paths in the following way. One path will be the portion of γ'^* that from z exits $B_z(R)$ in the direction of x . The other will be the portion of γ'^* from z to y pasted together with $\gamma_{y,2}^*$. This last path is, by definition, not \mathbb{L}^* -adjacent to $\gamma_{y,1}^*$, and cannot be \mathbb{L}^* -adjacent to the portion of γ'^* that coincides with γ''^* because of our assumption, in choosing y' to construct γ'^* , that this “touches” $\gamma_{y,1}^*$ before $\gamma_{y,2}^*$. \square

Remark 6.1. *Because of the monotonicity in the density of enhancement s , Lemmas 6.1 and 6.2 immediately imply the same conclusions for any stochastically enhanced configuration $\hat{\eta}$.*

6.1 The Critical Exponents

Proof of Lemma 4.1. The lower bound for $\theta(p, s)$ in equation (9) is obvious by monotonicity in s . For the upper bound, we rely on the following observation. Choose a positive constant K so that the annulus $B(R+K) \setminus B(R)$ contains at least one $*$ -loop. If no site in $B(R+K)$ is connected to infinity by an open \mathbb{L} -path before the enhancement takes place, then $B(R)$ must be surrounded by a closed, self-repelling $*$ -loop λ^* (i.e., $B(R) \subset \text{int}(\lambda^*)$). It then follows that each site in λ^* satisfies the hypotheses of Lemma 6.1. Therefore, the origin will not be connected to infinity by an open \mathbb{L} -path after the enhancement (see Remark 6.1).

For two subsets C and D of \mathbb{L} , we indicate with $\{C \longleftrightarrow D\}$ the event that some site in C is connected to some site in D by an open \mathbb{L} -path, with $\{C \longleftrightarrow \infty\}$ the event that some site in C belongs to an infinite, open \mathbb{L} -path. With this notation, we can write

$$\theta(p, s) \leq P_p(B(R+K) \longleftrightarrow \infty). \quad (18)$$

Since $\{o \longleftrightarrow \infty\} \supset \{y \text{ is open } \forall y \in B(R+K)\} \cap \{B(R+K) \longleftrightarrow \infty\}$, using the FKG inequality we have

$$P_p(o \longleftrightarrow \infty) \geq p^c P_p(B(R+K) \longleftrightarrow \infty), \quad (19)$$

with $c = \|B(R+K) \cap \mathbb{L}\|$. From this and (18), we get

$$\theta(p, s) \leq c_1 \theta(p), \quad (20)$$

with $c_1 = (1/p_c)^c$.

The lower bound for $\tau_{p,s}(x)$ in equation (10) is again obvious. To obtain the upper bound, we consider a site x at distance larger than $2(R+K)$ from the origin (where K is the constant introduced above). Unless $\{B(R+K) \longleftrightarrow B_x(R+K)\}$ before the enhancement takes place, $B(R)$ and $B_x(R)$ must be separated by a closed $*$ -loop surrounding one of them or by a doubly-infinite, closed $*$ -path. Therefore, by an application of Lemma 6.1, as before, it cannot be the case that $\{o \longleftrightarrow x\}$ after the enhancement. This yields

$$\tau_{p,s}(x) \leq P_p(B(R+K) \longleftrightarrow B_x(R+K)). \quad (21)$$

Since $\{o \longleftrightarrow x\} \supset \{y \text{ is open } \forall y \in B(R+K)\} \cap \{z \text{ is open } \forall z \in B_x(R+K)\} \cap \{B(R+K) \longleftrightarrow B_x(R+K)\}$, using the FKG inequality we have

$$P_p(o \longleftrightarrow x) \geq p^{c_2} P_p(B(R+K) \longleftrightarrow B_x(R+K)), \quad (22)$$

where $c_2 = 2c$ is independent of x and p . From this and (21), we get

$$\tau_{p,s}(x) \leq p^{-c_2} \tau_p(x), \quad (23)$$

as required.

Equation (11) is an immediate consequence of equation (10); it is enough to observe that

$$\lim_{|x| \rightarrow \infty} \left\{ -\frac{1}{|x|} [\log \tau_p(x) - c_2 \log p] \right\} = \xi(p)^{-1}. \quad \square \quad (24)$$

Proof of Theorems 1 and 6. It follows from (9) and (10) that

$$-\frac{\log \theta(p)}{\log(p - p_c)} \leq -\frac{\log \theta(p, s)}{\log(p - p_c)} \leq -\frac{\log c_1 + \log \theta(p)}{\log(p - p_c)}, \quad \text{for } p \in (p_c, 1], \quad (25)$$

$$\frac{\log \tau_{p_c}(x)}{\log |x|} \leq \frac{\log \tau_{p_c, s}(x)}{\log |x|} \leq \frac{\log \tau_{p_c}(x) - c_2 \log p}{\log |x|}, \quad \text{for } |x| \text{ large enough.} \quad (26)$$

For $p \in (0, p_c)$, observing that $\chi(p) = E_p \sum_{x \in \mathbb{L}} I_{\{o \longleftrightarrow x\}} = \sum_{x \in \mathbb{L}} \tau_p(x)$ (where $I_{\{\cdot\}}$ is the indicator function), (10) yields $\chi(p) \leq \chi(p, s) \leq p^{-c_2} \chi(p)$, and therefore

$$-\frac{\log \chi(p)}{\log(p - p_c)} \leq -\frac{\log \chi(p, s)}{\log(p - p_c)} \leq -\frac{\log \chi(p) - c_2 \log p}{\log(p - p_c)}. \quad (27)$$

Using these three equations, together with equation (11) and the definitions of the critical exponents, and taking the appropriate limits concludes the proof. \square

6.2 Crossing Probabilities

Proof of Theorems 2 and 7. We begin with a definition that will be useful in the proof. For (x, x') an ordered pair of \mathbb{L} -neighbors, we define the **partial cluster** $C_{(x, x')}$ to be the set of sites $y \in \mathbb{L}$ such that there is an \mathbb{L} -path $(x_0 = x', e_{x_0, x_1}, x_1, \dots, x_k = y)$ with $x_1 \neq x$ whose sites are all open or all closed.

To prove the theorem we need to compare crossing probabilities in $\hat{\eta}$ with crossing probabilities in η . In order to do that, we will use the natural coupling that exists between η and $\hat{\eta}$ via the enhancement. First of all notice that, if an open vertical crossing of \mathcal{R} is present in η , it is also present in $\hat{\eta}$. Therefore, recalling that $\varphi_\delta(b, h)$ (resp., $\hat{\varphi}_\delta(b, h)$) is the probability of an open vertical \mathbb{L} -crossing of \mathcal{R} from η (resp., $\hat{\eta}$), we have

$$\liminf_{\delta \rightarrow 0} \hat{\varphi}_\delta(b, h) \geq \lim_{\delta \rightarrow 0} \varphi_\delta(b, h) = F(\rho). \quad (28)$$

On the other hand, if an open vertical \mathbb{L} -crossing of \mathcal{R} is not present in η , this implies the existence of a closed horizontal \mathbb{L}^* -crossing of \mathcal{R} . For δ small, such a crossing must involve many sites, and the probability of finding “near” its endpoints two sites, x and y , belonging to the crossing and attached through closed paths to two protected sites, x' and y' , must be close to one. If such protected sites are found, it follows from Lemma 6.2 (and Remark 6.1) that, when δ is small enough, at least the portion of the closed horizontal crossing from x to y is still present in $\hat{\eta}$. This suggests that, conditioned on having in η a closed horizontal $*$ -crossing of a slightly bigger (in the horizontal direction) rectangle,

with high probability there will be in $\hat{\eta}$ a closed horizontal $*$ -crossing of \mathcal{R} blocking any open vertical \mathbb{L} -crossing. It is then enough to prove that this probability goes to one as $\delta \rightarrow 0$.

We will now make this more precise, adapting the proof of Theorem 1 of [15]. Consider the rectangle $\mathcal{R}' = \mathcal{R}(b', h)$ with b' slightly larger than b and aspect ratio $\rho' = b'/h$, and let $\psi_\delta^*(b', h)$ be the probability of a closed horizontal $*$ -crossing of $\mathcal{R}(b', h)$ from η . The presence of a closed horizontal $*$ -crossing of $\mathcal{R}(b', h)$ prevents any open vertical \mathbb{L} -crossing of $\mathcal{R}(b', h)$ and vice versa. Therefore,

$$\lim_{\delta \rightarrow 0} \varphi_\delta(b, h) \leq \lim_{\delta \rightarrow 0} (1 - \psi_\delta^*(b', h)) = F(b'/h). \quad (29)$$

Since, by continuity,

$$\lim_{b' \rightarrow b} F(b'/h) = F(\rho), \quad (30)$$

if we could replace $\lim_{\delta \rightarrow 0} \varphi_\delta(b, h)$ with $\limsup_{\delta \rightarrow 0} \hat{\varphi}_\delta(b, h)$ in (29), we would be done.

This is achieved by showing that if there is a closed horizontal $*$ -crossing $\gamma^* = (y_0, \dots, y_k)$ of \mathcal{R}' in η , then there is a closed horizontal $*$ -crossing of \mathcal{R} in $\hat{\eta}$ with probability going to one as $\delta \rightarrow 0$.

To do this, we take a b'' between b and b' and consider the rectangle $\mathcal{R}(b'', h)$. Assume that γ^* is parametrized from left to right and let y_{k_1+1} be the first site of γ^* outside of $\mathcal{R}'' \setminus \mathcal{R}$. Analogously, let y_{k_2-1} be the first site of γ^* outside of $\mathcal{R}'' \setminus \mathcal{R}$ when γ^* is parametrized in reversed order, from right to left. If we can find two protected sites, x_1 and x_2 , one in each of the partial clusters $C_{(y_{k_1+1}, y_{k_1})}$ and $C_{(y_{k_2-1}, y_{k_2})}$, contained inside $\mathcal{R}' \setminus \mathcal{R}''$, then we can use Lemma 6.2 to conclude that there is a closed $*$ -path from x_1 to x_2 in $\hat{\eta}$. It also follows from the proof of the lemma that, for δ small enough compared to $b'' - b'$, that path contains a subpath of the portion of γ^* inside \mathcal{R} , providing a closed horizontal crossing of \mathcal{R} in $\hat{\eta}$.

Let A be the event that there is no protected site in either the portion of $C_{(y_{k_1+1}, y_{k_1})}$ contained inside $\mathcal{R}' \setminus \mathcal{R}''$ or the portion of $C_{(y_{k_2-1}, y_{k_2})}$ contained inside $\mathcal{R}' \setminus \mathcal{R}''$. To conclude the proof, we need to show that the probability of A goes to zero as $\delta \rightarrow 0$. To do so, we first partition $\mathbb{L} = \mathbb{Z}^2, \mathbb{T}$ or \mathbb{H} into disjoint regions Q_i , as explained below, and denote by \mathcal{Q} the collection of these regions. We consider specific embeddings for the three lattices, corresponding to the only three possible regular tessellations of the plane (see, e.g., [4]). In the corresponding embeddings, the square lattice can be partitioned into squares, and the hexagonal and triangular lattices into hexagonal regions (this can be maybe better understood by looking at the dual lattices and interpreting each face as a site of the original lattice – see, for example, Figure 4 for how to partition the triangular lattice using the hexagonal one). The regions Q_i must be chosen large enough (depending on R) so that there is positive probability for a region to contain at least one protected site.

We now do an algorithmic construction (a related algorithmic construction is described in [24]) of the portions of the partial clusters $C_{(y_{k_1+1}, y_{k_1})}$ and $C_{(y_{k_2-1}, y_{k_2})}$ contained in $\mathcal{R}' \setminus \mathcal{R}''$. We describe briefly how to do this for $C_{(y_{k_1+1}, y_{k_1})}$ (the construction for $C_{(y_{k_2-1}, y_{k_2})}$ is the same). The idea is that one starts by setting $C_{(y_{k_1+1}, y_{k_1})}$ equal to y_{k_1} , then looks at y_{k_1} 's neighbors contained in $\mathcal{R}' \setminus \mathcal{R}''$, and adds to $C_{(y_{k_1+1}, y_{k_1})}$ those neighbors that

are closed. Then one proceeds by looking at y_{k_1} 's next-nearest neighbors contained in $\mathcal{R}' \setminus \mathcal{R}''$, and so on. This, however, is done in such a way that, when the first site in a region Q_i from \mathcal{Q} is checked and found to be closed, then the other sites in that region are checked next (in some deterministic order), before moving to a neighboring region. The construction stops when all of the portion of $C_{(y_{k_1+1}, y_{k_1})}$ contained in $\mathcal{R}' \setminus \mathcal{R}''$ has been “discovered” (together with its boundary of open sites).

Notice that the portions of the partial clusters $C_{(y_{k_1+1}, y_{k_1})}$ and $C_{(y_{k_2-1}, y_{k_2})}$ inside $\mathcal{R}' \setminus \mathcal{R}''$ contain at least a number of sites of the order of $(b'' - b')/\delta$, since they contain the paths (y_0, \dots, y_{k_1}) and (y_{k_2}, \dots, y_k) respectively. Therefore, if b', b'' and the size of the Q_i 's are fixed, since each Q_i contains a protected site with positive probability, the algorithmic construction shows that $P_{p_c}(A) \leq \exp(-c(b' - b'')/\delta)$, for some $c > 0$. \square

6.3 The Full Scaling Limit

To begin with, we need two preliminary results, the first one of which is a consequence of the fact that before the enhancement we are dealing with a Bernoulli product measure P_p and is valid for all $p \in (0, 1)$. In the next lemma (and elsewhere), the **diameter** $\text{diam}(\cdot)$ of a subset of \mathbb{R}^2 is defined as the maximal Euclidean distance between any two points of that subset.

Lemma 6.3. *Let $(x, y)_d$ be any (deterministic) dual edge; then for M large enough and for some constant $c > 0$,*

$$P_p(\exists \vec{\gamma}_d \ni (x, y)_d : \text{diam}(\vec{\gamma}_d) \geq M \text{ and } \vec{\gamma}_d \text{ does not contain a stable edge}) \leq e^{-cM}. \quad (31)$$

Proof. The proof requires partitioning the lattice \mathbb{H} , dual of \mathbb{T} , into identical regions Q_i and performing an algorithmic construction of $\vec{\gamma}_d$, starting from $(x, y)_d$, as a percolation exploration process, but with the additional rule that, when the exploration process enters a Q_i for the first time, all the sites of \mathbb{T} inside Q_i are checked next, according to some deterministic order. The regions Q_i can be constructed iteratively as explained in Figure 4; their size will depend on the range R of the enhancement, and has to be chosen large enough so that, whenever the exploration process enters an unexplored region Q_i , there is a strictly positive probability, bounded away from zero by a constant that does not depend on the past history of the exploration process, that a stable (dual) edge belonging to $\vec{\gamma}_d$ is found inside Q_i . Let F_i denote such an event.

To see that the probability of F_i can be bounded below by a positive constant, notice first that, because of the geometry of the Q_i 's, from every entrance point of a Q_i , there is a choice of the values (open or closed) of the sites of \mathbb{T} in the outermost layer of Q_i that forces the exploration process to enter that region, regardless of the past history of the exploration process. Moreover, each new region that the exploration process enters is “virgin” territory, on which no information is available.

If K is the number of dual edges contained in each region Q_i , then clearly the exploration process has to visit at least M/K regions Q_i , at each new visit having a chance of “bumping into” a stable edge. The bound in Lemma 6.3 follows immediately when M is

large enough compared to K . \square

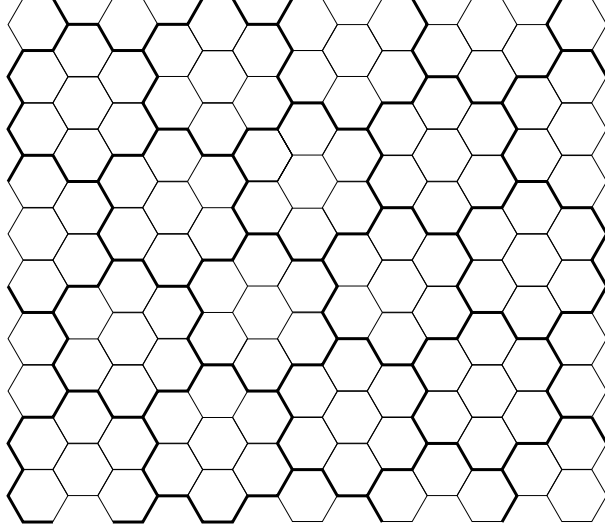


Figure 4: Partition of the hexagonal lattice into hexagonal cells. Each hexagon represents a site of \mathbb{T} . The process can be repeated iteratively, using the new cells instead of the original hexagons, to obtain a partition made of larger cells.

At criticality, any b-path is almost surely part of a complete boundary which forms a b-loop. The next lemma identifies a large “ancestor” $\tilde{\gamma}'_d \subset \Gamma(\eta)$ for each large enough b-loop $\tilde{\gamma}_d \subset \Gamma(\hat{\eta})$.

Lemma 6.4. *Set $\delta = 1$ for simplicity; then there is a one to one mapping from b-loops $\tilde{\gamma}_d \in \Gamma(\hat{\eta})$ with $\text{diam}(\tilde{\gamma}_d) \geq 6R$ to parent b-loops $\tilde{\gamma}'_d \in \Gamma(\eta)$ such that $\text{diam}(\tilde{\gamma}'_d) \geq \text{diam}(\tilde{\gamma}_d) - 4R$.*

Proof. First of all we remind the reader that in this case $\mathbb{L} = \mathbb{L}^* = \mathbb{T}$, so that \mathbb{L} -paths and $*$ -paths are the same. If the b-loop $\tilde{\gamma}_d$ is the external boundary of a closed cluster, such a cluster must come from a parent cluster in η whose diameter is equal to or bigger than that of the cluster in $\hat{\eta}$, since a closed cluster can only shrink. The external boundary $\tilde{\gamma}'_d$ of the closed cluster from η is then taken to be $\tilde{\gamma}_d$ ’s parent b-loop. To establish the one to one correspondence in this case, we have to show that two b-loops from $\hat{\eta}$ of diameter at least $6R$ cannot have the same parent b-loop in η . This can only happen if a closed cluster C from η of diameter larger than $6R$ splits in $\hat{\eta}$ into two closed clusters, C_1 and C_2 , each of diameter larger than $6R$. This implies that we can find a \mathbb{T} -path γ_0 , joining C_1 with C_2 and completely contained in C , such that, $\forall x \in \gamma_0$, $\eta(x) = 0$ and $\hat{\eta}(x) = 1$. However, γ_0 can be extended by two disjoint, nonadjacent \mathbb{T} -paths, γ_1 and γ_2 , of diameter larger than R , with γ_1 contained in C_1 and γ_2 contained in C_2 . This shows that each site $x \in \gamma_0$ satisfies the conditions of Lemma 6.1, so that $\hat{\eta}(x) = 0$, leading to a contradiction.

If the b-loop $\vec{\gamma}_d$ is the external boundary of an open cluster C from $\hat{\eta}$, consider two portions, $\vec{\gamma}_{d,1}$ and $\vec{\gamma}_{d,2}$, of $\vec{\gamma}_d$ such that $R \leq \text{diam}(\vec{\gamma}_{d,1}), \text{diam}(\vec{\gamma}_{d,2}) \leq 2R$ and $\sup_{u \in \vec{\gamma}_{d,1}, v \in \vec{\gamma}_{d,2}} |u - v| = \text{diam}(\vec{\gamma}_d)$ (so that $\inf_{u \in \vec{\gamma}_{d,1}, v \in \vec{\gamma}_{d,2}} |u - v| \geq \text{diam}(\vec{\gamma}_d) - 4R$). Call $\vec{\gamma}_{d,3}$ and $\vec{\gamma}_{d,4}$ the two remaining portions of $\vec{\gamma}_d$. In η , one of the two following statements must be true: either (1) $\vec{\gamma}_{d,1}$ and $\vec{\gamma}_{d,2}$ are connected by an open \mathbb{T} -path, or (2) $\vec{\gamma}_{d,3}$ and $\vec{\gamma}_{d,4}$ are connected by a closed \mathbb{T} -path. If (1) happens, then the path connecting $\vec{\gamma}_{d,1}$ and $\vec{\gamma}_{d,2}$ is the “backbone” of an open cluster whose external boundary we take as $\vec{\gamma}'_d$. The inequality $\text{diam}(\vec{\gamma}'_d) \geq \text{diam}(\vec{\gamma}_d) - 4R$ is clearly satisfied. The uniqueness of $\vec{\gamma}'_d$ comes from the fact that two open clusters from η of diameter at least $2R$ cannot merge, since each one of them is surrounded by a closed, self-repelling \mathbb{T} -loop (of diameter at least $2R$) such that each site of the loop satisfies the conditions in Lemma 6.1 and is therefore closed in $\hat{\eta}$.

To conclude the proof, we show by contradiction that (2) cannot happen. The open cluster C of which $\vec{\gamma}_d$ is the external boundary is surrounded (in $\hat{\eta}$) by a closed \mathbb{T} -loop λ , with every dual edge in $\vec{\gamma}_d$ separating a site of C from one in λ . Call λ_3 and λ_4 the portions of λ corresponding respectively to $\vec{\gamma}_{d,3}$ and $\vec{\gamma}_{d,4}$, in the sense that the sites of λ_3 and λ_4 are next to edges of $\vec{\gamma}_{d,3}$ and $\vec{\gamma}_{d,4}$. Notice that λ_3 and λ_4 are closed (in $\hat{\eta}$ and therefore also in η) \mathbb{T} -paths and that they are not \mathbb{T} -adjacent, since otherwise the set C would not be \mathbb{T} -connected.

Suppose that (2) did happen, then there would be a \mathbb{T} -path γ' , with $\eta(x) = 0 \ \forall x \in \gamma'$, contained in $\text{int}(\vec{\gamma}_d)$ and connecting λ_3 with λ_4 . From γ' one can extract a self-repelling \mathbb{T} -path γ joining λ_3 with λ_4 and with the property that it is not \mathbb{T} -adjacent to λ_3 or λ_4 if one excludes its first and last sites (this can be done by taking a minimal \mathbb{T} -path joining λ_3 with λ_4). Every site $x \in \gamma$ is then connected by two self-repelling, closed \mathbb{T} -paths to λ_3 and λ_4 , using which the two paths can be continued until they exit $B_x(R)$. If we exclude x , the two resulting paths are disjoint and are not \mathbb{T} -adjacent, because of the assumptions on $\vec{\gamma}_d$, $\vec{\gamma}_{d,1}$, and $\vec{\gamma}_{d,2}$. Thus, by Lemma 6.1, $\hat{\eta}(x) = 0$, $\forall x \in \gamma$, contradicting the assumption that $\vec{\gamma}_d$ is the external boundary of an open \mathbb{T} -cluster from $\hat{\eta}$. \square

Proof of Lemma 4.2. We can finally proceed to the proof of Lemma 4.2 itself. Let us start, for simplicity, with the case of a single “large” b-loop $\vec{\gamma}_d$ from η of diameter at least $\varepsilon/2$ for some fixed $\varepsilon > 0$. In the case of a “large” loop $\vec{\gamma}_d$, as $\delta \rightarrow 0$, we can apply Lemma 6.4 to obtain a daughter loop $\vec{\gamma}'_d$. We then have to show that for appropriate parametrizations g and g' of $\vec{\gamma}_d$ and $\vec{\gamma}'_d$, and for δ small enough,

$$\sup_{t \in [0,1]} d(g(t), g'(t)) < \varepsilon + 2\delta. \quad (32)$$

Once this is done, to prove the other direction for a single curve (i.e., given $\vec{\gamma}'_d \in \hat{\mathcal{F}}_\delta$ and g' , we need to find $\vec{\gamma}_d \in \mathcal{F}_\delta$ and g so that (32), in which the dependence on the scale factor δ has been suppressed, is valid), we use Lemma 6.4, which identifies a large parent b-loop in $\Gamma(\eta)$ for each large b-loop in $\Gamma(\hat{\eta})$. Later we will require that both directions hold simultaneously for all the loops (“large” and “small”) in \mathcal{F}_δ and $\hat{\mathcal{F}}_\delta$, as implied by (15).

For a given $\varepsilon > 0$, we divide \mathbb{R}^2 into two regions: $B(6/\varepsilon)$ and $\mathbb{R}^2 \setminus B(6/\varepsilon)$. We start

by showing that, thanks to the choice of the metric (12), one only has to worry about curves (or polygonal paths) that intersect $B(6/\varepsilon)$. Indeed, the distance between any two points $u, v \in \mathbb{R}^2 \setminus B(6/\varepsilon)$ satisfies the following bound

$$d(u, v) \leq d(u, \infty) + d(v, \infty) \leq 2 \int_0^\infty [1 + (l + 6/\varepsilon)^2]^{-1} dl < \varepsilon/3. \quad (33)$$

Thus, given any curve in \mathcal{F}_δ contained completely in $\mathbb{R}^2 \setminus B(6/\varepsilon)$, it can be approximated by any curve in $\hat{\mathcal{F}}_\delta$ also contained in $\mathbb{R}^2 \setminus B(6/\varepsilon)$, and vice versa. The existence of such curves in \mathcal{F}_δ is clearly not a problem, since the region $\mathbb{R}^2 \setminus B(6/\varepsilon)$ contains an infinite subset of $\delta\mathbb{H}$ and therefore there is zero probability that it doesn't contain any b-path in η . There is also zero probability that it contains no stable edge in η , but any such edge also belongs to $\hat{\mathcal{F}}_\delta$. Therefore, in the rest of the proof we consider only b-paths that intersect $B(6/\varepsilon)$.

Given a b-path $\vec{\gamma}_d = \{\xi_0, e_1^d, \dots, e_k^d, \xi_k\}$ in \mathcal{F}_δ with parametrization $g(t)$, we let $u_0 = \xi_0$. We will indicate by $\vec{\gamma}_d(u, v)$, with $u, v \in \vec{\gamma}_d$, the portion of $\vec{\gamma}_d$ between u and v . The following algorithmic construction produces a sequence u_0, \dots, u_N of points in $\vec{\gamma}_d$.

1. Start with u_0 .
2. Once u_0, \dots, u_i have been constructed, if $u_i \in B(6/\varepsilon)$, let u_{i+1} be the first intersection of $\vec{\gamma}_d \setminus \vec{\gamma}_d(u_0, u_i)$ with $\partial B_{u_i}(\varepsilon/3)$, if $u_i \notin B(6/\varepsilon)$, let u_{i+1} be the first intersection of $\vec{\gamma}_d \setminus \vec{\gamma}_d(u_0, u_i)$ with $\partial B_{u_i}^d(\varepsilon/3)$.
3. Terminate when there is no next u_i .

The algorithm stops after a finite number, N , of steps. During the construction of the sequence u_0, \dots, u_N , $\vec{\gamma}_d$ is split in $N + 1$ pieces, the first N having diameter at least $\varepsilon/3$. The construction also produces a sequence of balls $B_{u_i}(\varepsilon/3)$ or $B_{u_i}^d(\varepsilon/3)$, $i = 0, \dots, N$. Notice that no two successive u_i 's can lie outside of $B(6/\varepsilon)$. In fact, if for some i , u_i lies outside of $B(6/\varepsilon)$, u_{i+1} belongs to $\partial B_{u_i}^d(\varepsilon/3)$, which is contained inside $B(6/\varepsilon)$, due to the choice of the metric (12). Each u_i lies on a site or an edge of $\delta\mathbb{H}$, but no more than one u_i can lie on the same site or edge since $\vec{\gamma}_d$ is self-avoiding and cannot use the same site or edge more than once. Therefore the number of u_i 's in $B(6/\varepsilon)$ is bounded by a constant times the number of sites and edges contained in $B(6/\varepsilon)$. Also, the number of u_i 's lying outside of $B(6/\varepsilon)$ cannot be larger than (one plus) the number of the u_i 's lying inside $B(6/\varepsilon)$. Therefore, $N \leq \text{const} \times (\varepsilon\delta)^{-2}$.

For each $i = 0, \dots, N-1$, let $O_i = V_i \cup V_{i+1}$, where V_i is $B_{u_i}(\varepsilon/3 + \delta)$ if $u_i \in B(6/\varepsilon)$ and $B_{u_i}^d(\varepsilon/3 + \delta)$ if $u_i \notin B(6/\varepsilon)$. Now let v_i be the first intersection of $\vec{\gamma}_d(u_i, u_{i+1})$ with $B_{u_i}(\varepsilon/9)$ and assume that there exists a sequence $\bar{e}_0^d, \dots, \bar{e}_{N-1}^d$ of stable edges (see Definition 6.2) of $\vec{\gamma}_d$ with \bar{e}_i^d contained in $\vec{\gamma}_d(u_i, v_i)$. $\vec{\gamma}_d(\bar{e}_i^d, \bar{e}_{i+1}^d)$ is contained in O_i (for fixed ε and small enough δ). Besides, for fixed ε and small enough δ , any two successive stable edges $\bar{e}_i^d, \bar{e}_{i+1}^d$ lie next to two protected sites that satisfy the conditions in Lemma 6.2, where the closed $*$ -path γ^* can be taken to be a self-repelling path of closed sites along the b-path $\vec{\gamma}_d(\bar{e}_i^d, \bar{e}_{i+1}^d)$ and is therefore contained in O_i . As can be seen from the proof of the lemma,

the path γ'^* constructed from γ^* is also contained in O_i . γ'^* represents a barrier beyond which $\vec{\gamma}_d$ cannot move, since all of its sites are closed in $\hat{\eta}$, so that $\vec{\gamma}'_d(\bar{e}_i^d, \bar{e}_{i+1}^d)$ is confined to lie within $\vec{\gamma}_d(\bar{e}_i^d, \bar{e}_{i+1}^d)$ and γ'^* , and thus within O_i .

To parameterize $\vec{\gamma}'_d$, we use any parametrization $g'(t)$ such that $g'(t) = g(t)$ whenever $g(t) \in \bar{e}_i^d$. Using this parametrization and the previous fact, it is clear that the distance between $\vec{\gamma}_d(\bar{e}_i^d, \bar{e}_{i+1}^d)$ and $\vec{\gamma}'_d(\bar{e}_i^d, \bar{e}_{i+1}^d)$ does not exceed $\varepsilon + 2\delta$. Therefore, conditioning on the existence of the above sequence $\bar{e}_0^d, \dots, \bar{e}_{N-1}^d$ of stable edges of $\vec{\gamma}_d$, we can conclude that

$$\sup_{t \in [0,1]} d(g(t), g'(t)) < \varepsilon + 2\delta. \quad (34)$$

It remains to prove the existence of the sequence $\bar{e}_0^d, \dots, \bar{e}_{N-1}^d$ of stable edges. To do that, let us call A_i the event that $\vec{\gamma}_d(u_i, v_i)$ does *not* contain at least one stable edge, and let $A = \cup_{i=0}^{N-1} A_i$. Then, considering that the total number of dual edges contained in $B(6/\varepsilon)$ is bounded above by $\text{const} \times (\varepsilon\delta)^{-2}$ and using Lemma 6.3, we have

$$P_p(A) \leq (\varepsilon\delta)^{-2} e^{-c'(\varepsilon/\delta)} \quad (35)$$

for some $c' > 0$. Equation (35) means that the probability of not finding at least one stable edge in $\vec{\gamma}_d(u_i, v_i)$ for each $i = 0, \dots, N-1$ is very small and goes to 0, for fixed ε , as $\delta \rightarrow 0$. This is enough to conclude that, with high probability (going to 1 as $\delta \rightarrow 0$), equation (34) holds.

This proves one direction of the claim, in the case of a single curve. To obtain the other direction, as already explained, we use Lemma 6.4 which identifies a parent b-loop in $\Gamma(\eta)$ for each large b-loop in $\Gamma(\hat{\eta})$. Lemma 6.4 also insures that the parent b-loop is large enough so that we can apply to it the above arguments and obtain the desired result.

At this point, we need to show that the above argument can be repeated and the construction done simultaneously for all curves in \mathcal{F}_δ and $\hat{\mathcal{F}}_\delta$. First of all notice that for a fixed ε , any b-path $\vec{\gamma}_d$ of diameter less than $\varepsilon/2$ can be approximated by a closest stable edge, provided that one is found within a ball of radius $\varepsilon/2$ that contains $\vec{\gamma}_d$, with the probability of this last event clearly going to 1 as $\delta \rightarrow 0$, when we restrict attention to $B(6/\varepsilon)$. For a b-path outside $B(6/\varepsilon)$, we already noticed that it can be approximated by any other b-path also outside $B(6/\varepsilon)$. As for the remaining b-paths, notice that the total number of boundaries that intersect the ball $B(6/\varepsilon)$ cannot exceed $\text{const} \times (\varepsilon\delta)^{-2}$. Thus, we can carry out the above construction simultaneously for all the boundaries that touch $B(6/\varepsilon)$, having to deal with at most $\text{const} \times (\varepsilon\delta)^{-2}$ segments of b-paths of diameter at least $\varepsilon/2$. Therefore, letting $Y_\delta = \text{dist}(\mathcal{F}_\delta, \hat{\mathcal{F}}_\delta)$, we can apply once again Lemma 6.3 and conclude that

$$\mathbb{P}(Y_\delta > \varepsilon) \leq (\varepsilon\delta)^{-2} e^{-c''(\varepsilon/\delta)}, \quad (36)$$

for some $c'' > 0$.

To show that $Y_\delta \rightarrow 0$ \mathbb{P} -almost surely, as $\delta \rightarrow 0$, and thus conclude the proof, it suffices to show that, $\forall \varepsilon > 0$, $\mathbb{P}(\limsup_{\delta \rightarrow 0} Y_\delta > \varepsilon) = 0$. To that end, first take a sequence

$\delta_k = 1/2^k$ and notice that

$$\sum_{k=0}^{\infty} \mathbb{P}(Y_{\delta_k} > \varepsilon) \leq \sum_{k=0}^{\infty} \frac{4^k}{\varepsilon^2} e^{-c'' 2^k \varepsilon} < \infty, \quad (37)$$

where we have made use of (36). Equation (37) implies that we can apply the Borel-Cantelli lemma and deduce that $\mathbb{P}(\limsup_{k \rightarrow \infty} Y_{\delta_k} > \varepsilon) = 0, \forall \varepsilon > 0$. In order to handle the values of δ not in the sequence δ_k , that is those δ such that $\delta_{k+1} < \delta < \delta_k$ for some k , we use the double bound

$$a d(u, v) \leq d(au, av) \leq \frac{1}{a} d(u, v), \quad (38)$$

valid for any $0 < a < 1$, which implies that $a Y_{\delta_k} \leq Y_{a\delta_k} \leq \frac{1}{a} Y_{\delta_k}$. The two bounds in equation (38) come from writing $d(au, av)$ as $d(au, av) = \inf_{f'} \int (1 + |f'|^2)^{-1} dl' = a \inf_f \int (1 + a^2 |f|^2)^{-1} dl$, where $f'(l')$ are smooth curves joining au with av , while $f(l)$ are smooth curves joining u with v . The proof of the theorem is now complete. \square

In order to prove Theorem 3, we will use the following general fact, of which we include a proof for completeness.

Lemma 6.5. *If $\{X_\delta\}, \{Y_\delta\}$ (for $\delta > 0$), and X are random variables taking values in a complete, separable metric space S (whose σ -algebra is the Borel algebra) with $\{X_\delta\}$ and $\{Y_\delta\}$ all defined on the same probability space, then if X_δ converges in distribution to X and the metric distance between X_δ and Y_δ tends to zero almost surely as $\delta \rightarrow 0$, Y_δ also converges in distribution to X .*

Proof. Since X_δ converges to X in distribution, the family $\{X_\delta\}$ is relatively compact and therefore tight by an application of Prohorov's Theorem (using the fact that S is a complete, separable metric space – see, e.g., [6]). Then, for any bounded, continuous, real function f on S , and for any $\varepsilon > 0$, there exists a compact set \mathcal{K} such that $\int |f(X_\delta)| I_{\{X_\delta \notin \mathcal{K}\}} dP < \varepsilon$ and $\int |f(Y_\delta)| I_{\{X_\delta \notin \mathcal{K}\}} dP < \varepsilon$ for all δ , where $I_{\{\cdot\}}$ is the indicator function and P the probability measure of the probability space of $\{X_\delta\}$ and $\{Y_\delta\}$. Thus, for small enough δ ,

$$\left| \int f(X_\delta) dP - \int f(Y_\delta) dP \right| < \int |f(X_\delta) - f(Y_\delta)| I_{\{X_\delta \in \mathcal{K}\}} dP + 2\varepsilon < 3\varepsilon, \quad (39)$$

where in the last inequality we use the absolute continuity of f when restricted to the compact set \mathcal{K} and the fact that the metric distance between X_δ and Y_δ goes to 0 as $\delta \rightarrow 0$. \square

Proof of Theorem 3. The theorem is an immediate consequence of Lemma 4.2 and Lemma 6.5. It is enough to apply Lemma 6.5 to the triplet $\mu_\delta, \hat{\mu}_\delta, \mu$ (or, to be more precise, to the random variables of which those are the distributions), where μ is the full scaling limit of critical site percolation on the triangular lattice [12, 13]. \square

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Appendix A: Matching Pairs of Lattices

The proofs of the results of this paper concerning enhancement percolation and the universality of critical exponents apply not only to the square, triangular and hexagonal lattice, but to the class of regular lattices considered in [29]. These are infinite periodic graphs embedded in some suitable way in \mathbb{R}^2 . While we refer to Chapter 2 of [29] for the relevant definitions not explicitly given here and all the details, we explain below the general notion of matching pairs of graphs (or lattices). Within this general framework, which includes in particular the three lattices we are mainly interested in, we then provide the proofs of the “geometric” lemmas needed in Section 5 in the the proof of Lemma 3.1.

Let \mathbb{M} be a regular, planar lattice embedded in \mathbb{R}^2 (a **mosaic** in the language of Sykes and Essam [41], and Kesten [29]) such that each component F of $\mathbb{R}^2 \setminus \mathbb{M}$ is bounded by a Jordan curve (i.e., a simple, closed curve) made up of a finite number of edges of \mathbb{M} . F is called a **face** of \mathbb{M} , the edges delimiting F form its **perimeter**, and the sites of \mathbb{M} in the perimeter of F are its **vertices**, which we denote by $V(F)$. **Close-packing** a face F of \mathbb{M} means adding an edge between each pair of vertices of F that do not already share an edge.

Given a mosaic \mathbb{M} and a subset \mathfrak{F} of its collection of faces, a lattice \mathbb{L} is obtained from \mathbb{M} by close-packing all the faces in \mathfrak{F} , and \mathbb{L}^* by close-packing all the faces not in \mathfrak{F} . The pair $(\mathbb{L}, \mathbb{L}^*)$ is a **matching pair of lattices**.

In the embedding of \mathbb{L} and \mathbb{L}^* one can choose to draw the edges added to \mathbb{M} when close-packing a face F inside that same face. In a matching pair usually at least one of the lattices \mathbb{L} or \mathbb{L}^* is not planar. Notice that $\mathfrak{F} = \emptyset$ is allowed in the previous definition; in that case \mathbb{L} coincides with \mathbb{M} and \mathbb{L}^* is the close-packed version of \mathbb{M} .

The definitions of \mathbb{L} - and \mathbb{L}^* -adjacent, \mathbb{L} - and $*$ -path, \mathbb{L} - and $*$ -connected, and external (site) boundary are the same as in Section 3.1. We will also use the same notation for sites and paths as in that section.

It is important to observe (see, for example, [41] and Corollary 2.2 of [29]) that the external (site) boundary of a nonempty, bounded, \mathbb{L} -connected set C of sites of \mathbb{L} forms, together with the edges between sites in the boundary, a self-avoiding $*$ -loop λ^* such that all the sites in C belong to $\text{int}(\lambda^*)$.

Note that the square, triangular and hexagonal lattices considered in the main part of the paper are of the type described above with $\mathbb{L} = \mathbb{M}$.

Lemma A.1. *Given a self-repelling \ast -loop λ^\ast and an edge $e = (x, y)$ in λ^\ast : (i) if e belongs to the perimeters of faces F_1 and F_2 of \mathbb{M} , then F_1 and F_2 belong one to $\text{int}(\lambda^\ast)$ and the other to $\text{ext}(\lambda^\ast)$; (ii) if e is contained in the interior of face $F \notin \mathfrak{F}$, then the two portions of F separated by e belong one to $\text{int}(\lambda^\ast)$ and the other to $\text{ext}(\lambda^\ast)$.*

Proof. (i) Since λ^\ast is self-repelling, sites in the perimeter of F_1 (resp., F_2) other than x and y can belong to λ^\ast only if F_1 (resp., F_2) is not close-packed in \mathbb{L}^\ast . This implies that λ^\ast does not cut through F_1 , nor F_2 . Therefore the two faces, being on opposite sides of λ^\ast , belong one to $\text{int}(\lambda^\ast)$ and the other to $\text{ext}(\lambda^\ast)$.

(ii) In this case, F is of necessity close-packed in \mathbb{L}^\ast and so no site in $V(F)$ other than x and y can be in λ^\ast . Therefore the two portions of F , being on opposite sides of λ^\ast , belong one to $\text{int}(\lambda^\ast)$ and the other to $\text{ext}(\lambda^\ast)$. \square

Lemma A.2. *If λ^\ast is a self-repelling \ast -loop such that $\text{int}(\lambda^\ast) \cap V(\mathbb{L})$ is not empty, then each site in λ^\ast has at least one \mathbb{L} -neighbor in $\text{int}(\lambda^\ast) \cap V(\mathbb{L})$ and one in $\text{ext}(\lambda^\ast) \cap V(\mathbb{L})$.*

Proof. Consider two consecutive sites of λ^\ast , z_{j-1} and z_j . We want to show that z_j has at least one neighbor in each of the two Jordan domains that make up $\mathbb{R}^2 \setminus \lambda^\ast$. We first assume that the edge (z_{j-1}, z_j) belongs to the perimeters of two faces of \mathbb{M} , F_1 and F_2 . By Lemma A.1, one of the two faces, say F_1 , must be in $\text{int}(\lambda^\ast)$ and the other one, F_2 , in $\text{ext}(\lambda^\ast)$.

If F_1 belongs to \mathfrak{F} , it is close-packed in \mathbb{L} and therefore z_j is \mathbb{L} -adjacent to all the vertices of F_1 . On the other hand, not all the vertices of F_1 can belong to λ^\ast , otherwise this would not be self-repelling. This shows that z_j has an \mathbb{L} -neighbor in $V(F_1)$ which is not in λ^\ast , thus it has an \mathbb{L} -neighbor in $\text{int}(\lambda^\ast)$. If F_1 does not belong to \mathfrak{F} , it is close-packed in \mathbb{L}^\ast and therefore the sites of $V(F_1)$ other than z_{j-1} and z_j do not belong to λ^\ast . Then, z_j has an \mathbb{L} -neighbor in $V(F_1)$ which does not belong to λ^\ast and thus belongs to $\text{int}(\lambda^\ast)$. Arguments analogous to the ones above, but with F_1 replaced by F_2 , show that z_j must have an \mathbb{L} -neighbor that belongs to $\text{ext}(\lambda^\ast)$.

If the edge (z_{j-1}, z_j) is contained in the interior of a face F , then of necessity F is close-packed in \mathbb{L}^\ast and no site of $V(F)$ other than z_{j-1} and z_j belongs to λ^\ast . By Lemma A.1, (z_{j-1}, z_j) splits F in two parts, one contained in $\text{int}(\lambda^\ast)$ and the other in $\text{ext}(\lambda^\ast)$. The perimeter of each of those two parts contains (z_{j-1}, z_j) plus at least two more edges and one site \mathbb{L} -adjacent to z_j . Therefore z_j has an \mathbb{L} -neighbor in $\text{int}(\lambda^\ast)$ and one in $\text{ext}(\lambda^\ast)$. \square

Lemma A.3. *A (finite) self-repelling \ast -loop λ^\ast partitions $\mathbb{L} \setminus \lambda^\ast$ into two \mathbb{L} -connected components, one bounded and the other unbounded.*

Remark A.1. *Note that this is a stronger statement than saying that λ^\ast , being a Jordan curve, partitions $\mathbb{R}^2 \setminus \lambda^\ast$ in two components. In fact, we are claiming that the subsets of \mathbb{L} contained in $\text{int}(\lambda^\ast)$ and $\text{ext}(\lambda^\ast)$ are \mathbb{L} -connected, and this can be false if λ^\ast is not self-repelling.*

Proof. As explained in Remark A.1, we need to prove that $\text{int}(\lambda^\ast) \cap \mathbb{L}$ and $\text{ext}(\lambda^\ast) \cap \mathbb{L}$ are \mathbb{L} -connected. We only give the proof of this fact for $\text{int}(\lambda^\ast) \cap \mathbb{L}$ (that for $\text{ext}(\lambda^\ast) \cap \mathbb{L}$ being

analogous) and split it in two parts. We will first show that, for each site $z_j \in \lambda^*$, any two sites belonging to the set $\mathcal{N}_{\mathbb{L}}^{\lambda^*}(z_j)$ of \mathbb{L} -neighbors of z_j contained in $\text{int}(\lambda^*)$ can be joined by an \mathbb{L} -path completely contained in $\text{int}(\lambda^*)$. Then, for any two successive sites z_{j-1} and z_j in λ^* , we will show that they have two neighbors, z'_{j-1} and z'_j respectively, which belong to $\text{int}(\lambda^*) \cap V(\mathbb{L})$ and such that there is an \mathbb{L} -path joining them and completely contained in $\text{int}(\lambda^*)$.

These two facts prove that any site in $\mathcal{N}_{\mathbb{L}}^{\lambda^*}(z_{j-1})$ can be joined to any site in $\mathcal{N}_{\mathbb{L}}^{\lambda^*}(z_j)$ by an \mathbb{L} -path completely contained in $\text{int}(\lambda^*)$. Therefore, for each pair of sites z_i and z_j in λ^* , any site in $\mathcal{N}_{\mathbb{L}}^{\lambda^*}(z_i)$ can be joined to any site in $\mathcal{N}_{\mathbb{L}}^{\lambda^*}(z_j)$ by an \mathbb{L} -path completely contained in $\text{int}(\lambda^*)$. This clearly proves the claim, since each $x \in \text{int}(\lambda^*) \cap V(\mathbb{L})$ is either an \mathbb{L} -neighbor of a site in λ^* or can be joined to one by an \mathbb{L} -path completely contained in $\text{int}(\lambda^*)$.

We will now prove the first claim about $\mathcal{N}_{\mathbb{L}}^{\lambda^*}(z_j)$, for $z_j \in \lambda^*$. By Lemma A.2, z_j has at least one \mathbb{L} -neighbor in $\text{int}(\lambda^*) \cap \mathbb{L}$. If it has exactly one such \mathbb{L} -neighbor, there is nothing to prove, so we assume that z_j has at least two \mathbb{L} -neighbors in $\text{int}(\lambda^*) \cap \mathbb{L}$. We first show that there is an \mathbb{L} -path, contained in $\text{int}(\lambda^*)$, joining any two elements of $\mathcal{N}_{\mathbb{L}}^{\lambda^*}(z_j)$ whenever they are vertices of the same face F . Consider two such sites x and y in $\mathcal{N}_{\mathbb{L}}^{\lambda^*}(z_j)$. If F is close-packed in \mathbb{L} , x and y are automatically adjacent in \mathbb{L} . If F is not close-packed in \mathbb{L} , x and y are the only \mathbb{L} -neighbors of z_j belonging to $V(F)$. Let z_{j-1} and z_{j+1} be the sites that come before and after z_j in λ^* . They cannot both belong to $V(F)$, otherwise λ^* would not be self-repelling. If only one, say z_{j-1} belongs to $V(F)$, then the edge (z_{j-1}, z_j) that cuts through F belongs to λ^* and, by Lemma A.1 (ii), it cannot be the case that x and y are both in $\text{int}(\lambda^*)$, contradicting our hypothesis. Thus, neither z_{j-1} nor z_{j+1} belongs to $V(F)$, which implies that, since F is close-packed in \mathbb{L}^* , no site of $V(F)$ other than z_j can belong to λ^* (or else this would not be self-repelling). Therefore, all the sites in $V(F) \setminus \{z_j\}$ are inside $\text{int}(\lambda^*)$ and can be used to join x with y with an \mathbb{L} -path contained in $\text{int}(\lambda^*)$.

If x and y are both in $\mathcal{N}_{\mathbb{L}}^{\lambda^*}(z_j)$ but they are not vertices of the same face, then we are going to prove that there exists a sequence $z_j^0, z_j^1, \dots, z_j^k$, with $z_j^0 = x$ and $z_j^k = y$, of elements of $\mathcal{N}_{\mathbb{L}}^{\lambda^*}(z_j)$ such that for each $i \in 1, \dots, k$ there is a face F_i of \mathbb{M} with $z_j^{i-1}, z_j^i \in V(F_i)$, so that we can use again the result just above to find an \mathbb{L} -path inside $\text{int}(\lambda^*)$ joining x with y .

To see this, consider all the edges of \mathbb{L}^* incident on z_j . Since λ^* is self-repelling, only two of those edges and the two $*$ -neighbors of z_j that they are incident on can belong to λ^* . Then, the remaining edges and $*$ -neighbors of z_j are divided in two groups, those that lie to the right of λ^* and those that lie to its left. The edges and sites in one group are contained in $\text{int}(\lambda^*)$, those in the other group are contained in $\text{ext}(\lambda^*)$. x and y belong to the group in $\text{int}(\lambda^*)$. Since we are assuming that x and y are not vertices of the same face, spanning the wedge from the edge (x, z_j) to the edge (y, z_j) , or vice versa, in such a way as to remain inside $\text{int}(\lambda^*)$, one must encounter one or more edges of \mathbb{M} incident on z_j . Assuming (without loss of generality) that the right way to span the wedge is from (x, z_j) to (y, z_j) , we order those edges according to the order in which they are encountered moving from (x, z_j) to (y, z_j) , including (x, z_j) and (y, z_j) , and call them, respectively,

$(z_j^0, z_j), (z_j^1, z_j), \dots, (z_j^k, z_j)$, with $z_j^0 = x$ and $z_j^k = y$. The sequence $z_j^0, z_j^1, \dots, z_j^k$ is what we are after, and with this the claim that any two sites in $\mathcal{N}_{\mathbb{L}}^{\lambda^*}(z_j)$ can be connected by an \mathbb{L} -path completely contained in $\text{int}(\lambda^*)$ is proved.

In order to conclude the proof, we will now show that, given z_{j-1} and z_j in λ^* , one can pick a site z'_{j-1} from $\mathcal{N}_{\mathbb{L}}^{\lambda^*}(z_{j-1})$ and a site z'_j from $\mathcal{N}_{\mathbb{L}}^{\lambda^*}(z_j)$ such that there is an \mathbb{L} -path joining z'_{j-1} and z'_j , and completely contained in $\text{int}(\lambda^*)$.

We first assume that the edge (z_{j-1}, z_j) belongs to the perimeters of two faces, F_1 and F_2 , of \mathbb{M} . By Lemma A.1, one of these two faces, say F_1 , is contained inside $\text{int}(\lambda^*)$. If F_1 belongs to \mathfrak{F} , it is close-packed in \mathbb{L} and therefore z_{j-1} and z_j are \mathbb{L} -adjacent to all the other vertices of F_1 . On the other hand, not all the vertices of F_1 can belong to λ^* , otherwise this would not be self-repelling. Thus, z_{j-1} and z_j have a common \mathbb{L} -neighbor in $V(F_1)$ which does not belong to λ^* and is therefore contained in $\text{int}(\lambda^*)$. This common neighbor is our choice for both z'_{j-1} and z'_j .

If F_1 does not belong to \mathfrak{F} , it is close-packed in \mathbb{L}^* and therefore the sites of $V(F_1)$ other than z_{j-1} and z_j do not belong to λ^* . Then, z_{j-1} and z_j have \mathbb{L} -neighbors z'_{j-1} and z'_j in $V(F_1)$ which do not belong to λ^* and are contained in $\text{int}(\lambda^*)$. z'_{j-1} and z'_j may coincide or be \mathbb{L} -adjacent; if not, they nonetheless belong to the perimeter of the same face F_1 and can therefore be joined by an \mathbb{L} -path contained in $\text{int}(\lambda^*)$ that uses the other sites of $V(F_1)$.

If the edge (z_{j-1}, z_j) is contained in the interior of a face F , then of necessity F is close-packed in \mathbb{L}^* and no site in $V(F)$ other than z_{j-1} and z_j belongs to λ^* . By Lemma A.1, (z_{j-1}, z_j) splits F in two parts, one contained in $\text{int}(\lambda^*)$ and the other in $\text{ext}(\lambda^*)$. The perimeter of the portion in $\text{int}(\lambda^*)$ contains (z_{j-1}, z_j) plus at least two more edges and two sites, z'_{j-1} and z'_j (which may coincide), \mathbb{L} -adjacent to z_{j-1} and z_j respectively. z'_{j-1} and z'_j either coincide, or are \mathbb{L} -adjacent, or can be joined by an \mathbb{L} -path that uses vertices of F in $\text{int}(\lambda^*)$. \square

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